GLOBAL CLASSICAL SOLUTIONS TO THE COMPRESSIBLE EULER-MAXWELL EQUATIONS

JIANG XU[*]

Abstract. In this paper, we consider the compressible Euler-Maxwell equations arising in semi-conductor physics, which take the form of Euler equations for the conservation laws of mass density and current density for electrons, coupled to Maxwell's equations for self-consistent electromagnetic field. We study the global well-posedness in critical spaces and the limit to zero of some physical parameters in the scaled Euler-Maxwell equations. More precisely, using high- and low-frequency decomposition methods, we first construct uniform (global) classical solutions (around constant equilibrium) to the Cauchy problem of Euler-Maxwell equations in Chemin-Lerner's spaces with critical regularity. Furthermore, based on Aubin-Lions compactness lemma, it is justified that the (scaled) classical solutions converge globally in time to the solutions of compressible Euler-Poisson equations in the process of non-relativistic limit and to that of drift-diffusion equations under the relaxation limit or the combined non-relativistic and relaxation limits.

Key words. Euler-Maxwell equations, classical solutions, Chemin-Lerner's spaces, non-relativistic limit, relaxation limit

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1. Introduction and main results. The increasing demand on semiconductor devices has led to the necessity of a deep and detailed understanding on the mathematical theory of various charge-carrier transport models. Of these important models, the classical hydrodynamic model (also named as the Euler-Poisson equations), which treats the propagation of electrons in semiconductor devices as the flow of a compressible charged fluid in an electric field, has received increasing attention. For the cases of high electric field and submicronic devices, the Euler-Poisson equations of fluid dynamical form can represent a reasonable comprise between physical accuracy and reduction of computational cost in real applications, the reader is referred to [22] for more explanations. When semiconductor devices are operated under some high frequency conditions (such as photoconductive switches, electro-optics, semiconductor lasers and high-speed computers), magnetic fields are generated by moving electrons inside devices, then the electrons transport interacts with the propagating electromagnetic waves. In this case, the transport process is typically governed by the Euler-Maxwell equations, which is more accurate than the Euler-Poisson equations, since the electromagnetic field obeys Maxwell's equations instead of Poisson equation for the electric field only.

After some appropriate re-scaling, the compressible Euler-Maxwell equations are written, in nondimensional form, as

$$(1.1) \quad \begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \\ \partial_t (n\mathbf{u}) + \nabla \cdot (n\mathbf{u} \otimes \mathbf{u}) + \nabla P(n) = -n(\mathbf{E} + \varepsilon \mathbf{u} \times \mathbf{B}) - \frac{n\mathbf{u}}{\tau}, \\ \varepsilon \lambda^2 \partial_t \mathbf{E} - \nabla \times \mathbf{B} = \varepsilon n\mathbf{u}, \\ \varepsilon \partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \\ \lambda^2 \nabla \cdot \mathbf{E} = \bar{n} - n, \quad \nabla \cdot \mathbf{B} = 0, \end{cases}$$

for $(t,x) \in [0,+\infty) \times \mathbf{R}^N (N \geq 2)$. Here the unknowns $n, \mathbf{u} = (u_1, u_2, \cdots, u_N)^\top$, $\mathbf{E} = (E_1, E_2, \cdots, E_N)^\top$, $\mathbf{B} = (B_1, B_2, \cdots, B_N)^\top$ ($\top$ transpose) denote the electron density,

[*]Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, P.R.China (`jiangxu_79@yahoo.com.cn`, `jiangxu_79@nuaa.edu.cn`).

electron velocity, electric field and magnetic field, respectively. The pressure $P(n)$ satisfies the usual γ-law

$$(1.2) \qquad\qquad P(n) = P_0 n^\gamma (\gamma \geq 1),$$

where $P_0 > 0$ is some physical constant. The system (1.1) is called *isentropic* if $\gamma > 1$ and *isothermal* if $\gamma = 1$. $\tau, \lambda > 0$ are the (scaled) constants for the momentum-relaxation time and the Debye length. $c = (\epsilon_0 \upsilon_0)^{-\frac{1}{2}} > 0$ is the speed of light, where ϵ_0 and υ_0 are the vacuum permittivity and permeability. Setting $\varepsilon = \frac{1}{c}$. The independent parameters τ, λ and ε which arise from nondimensionalization, are assumed to be very small compared to the reference physical size. The symbols ∇, $\cdot$, $\times$ and $\otimes$ are the gradient operator, the scalar products, the vector products and the tensor products of two vectors, respectively. $\bar{n} > 0$ is the doping profile, which stands for the density of positively charged background ions.

It is not difficult to see that the above Euler-Maxwell equations consist of a quasi-linear hyperbolic system, the main feature of which is the finite time blowup of classical solutions even when the initial data are smooth and small. Hence, the qualitative study and device simulation of (1.1) are far to be trivial. In this paper, our main aim is to establish the global well-posedness and justify some singular limits for the Cauchy problem. For this purpose, the Euler-Maxwell equations (1.1) are equipped with the following initial conditions for $n, \mathbf{u}, \mathbf{E}$ and $\mathbf{B}$:

$$(1.3) \qquad (n, \mathbf{u}, \mathbf{E}, \mathbf{B})(x, 0) = (n_0, \mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0)(x), \quad x \in \mathbf{R}^N,$$

which satisfies the compatible conditions

$$(1.4) \qquad \lambda^2 \nabla \cdot \mathbf{E}_0 = \bar{n} - n_0, \quad \nabla \cdot \mathbf{B}_0 = 0, \quad x \in \mathbf{R}^N.$$

1.1. Singular limit analysis. It is convenient to state previous works and main results of this paper, we first introduce some singular limits in the scaled Euler-Maxwell equations at the formal level, including the non-relativistic limit, relaxation limit as well as combined non-relativistic and relaxation limits.

Firstly, we observe the non-relativistic limit (*i.e.* $\varepsilon \to 0$). Let $\tau = 1 = \lambda$ and $(n^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{E}^\varepsilon, \mathbf{B}^\varepsilon)$ be the solution of the following equations

$$(1.5) \qquad \begin{cases} \partial_t n^\varepsilon + \nabla \cdot (n^\varepsilon \mathbf{u}^\varepsilon) = 0, \\ \partial_t (n^\varepsilon \mathbf{u}^\varepsilon) + \nabla \cdot (n^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \nabla P(n^\varepsilon) \\ \qquad\qquad = -n^\varepsilon (\mathbf{E}^\varepsilon + \varepsilon \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon) - n^\varepsilon \mathbf{u}^\varepsilon, \\ \varepsilon \partial_t \mathbf{E}^\varepsilon - \nabla \times \mathbf{B}^\varepsilon = \varepsilon n^\varepsilon \mathbf{u}^\varepsilon, \\ \varepsilon \partial_t \mathbf{B}^\varepsilon + \nabla \times \mathbf{E}^\varepsilon = 0, \\ \nabla \cdot \mathbf{E}^\varepsilon = \bar{n} - n^\varepsilon, \quad \nabla \cdot \mathbf{B}^\varepsilon = 0. \end{cases}$$

Formally, we see that the limits $n^0, \mathbf{u}^0, \mathbf{E}^0$ of $n^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{E}^\varepsilon$ as $\varepsilon \to 0$ satisfy

$$(1.6) \qquad \begin{cases} \partial_t n^0 + \nabla \cdot (n^0 \mathbf{u}^0) = 0, \\ \partial_t (n^0 \mathbf{u}^0) + \nabla \cdot (n^0 \mathbf{u}^0 \otimes \mathbf{u}^0) + \nabla P(n^0) = -n^0 \mathbf{E}^0 - n^0 \mathbf{u}^0, \\ \nabla \cdot \mathbf{E}^0 = \bar{n} - n^0, \quad \nabla \times \mathbf{E}^0 = 0, \end{cases}$$

which is the well-known Euler-Poisson equations for semiconductors. The irrotationality of $\mathbf{E}^0$ implies the existence of a potential function $\varPhi^0$ such that $\mathbf{E}^0 = -\nabla \varPhi^0$. Then using the Green's formulation, (1.6) can be reduced to the form of the conservation law with a non-local source term, e.g., see [13].

Secondly, we justify the relaxation limit (*i.e.* $\tau \to 0$) in the Euler-Maxwell equations (1.1). The diffusion limit was first introduced by Marcati and Natalini [21] for the Euler-Poisson equations (1.6). Set $\varepsilon = 1 = \lambda$. To do this, as in [21], we define the following scaled transform

$$(1.7) \qquad (n^\tau, \mathbf{u}^\tau, \mathbf{E}^\tau, \mathbf{B}^\tau)(t, x) = \left(n, \frac{1}{\tau}\mathbf{u}, \mathbf{E}, \mathbf{B}\right)\left(\frac{t}{\tau}, x\right).$$

Then the new variable $(n^\tau, \mathbf{u}^\tau, \mathbf{E}^\tau, \mathbf{B}^\tau)$ satisfies

$$(1.8) \qquad \begin{cases} \partial_t n^\tau + \nabla \cdot (n^\tau \mathbf{u}^\tau) = 0, \\ \tau^2 \partial_t(n^\tau \mathbf{u}^\tau) + \tau^2 \nabla \cdot (n^\tau \mathbf{u}^\tau \otimes \mathbf{u}^\tau) + \nabla P(n^\tau) \\ \qquad\qquad\qquad = -n^\tau(\mathbf{E}^\tau + \tau \mathbf{u}^\tau \times \mathbf{B}^\tau) - n^\tau \mathbf{u}^\tau, \\ \tau \partial_t \mathbf{E}^\tau - \nabla \times \mathbf{B}^\tau = \tau n^\tau \mathbf{u}^\tau, \\ \tau \partial_t \mathbf{B}^\tau + \nabla \times \mathbf{E}^\tau = 0, \\ \nabla \cdot \mathbf{E}^\tau = \bar{n} - n^\tau, \quad \nabla \cdot \mathbf{B}^\tau = 0. \end{cases}$$

Formally, the limits $\mathcal{N}, \mathcal{E}$ of $n^\tau, \mathbf{E}^\tau$ as $\tau \to 0$ satisfy the so-called drift-diffusion equations

$$(1.9) \qquad \begin{cases} \partial_t \mathcal{N} = \nabla \cdot (\nabla P(\mathcal{N}) + \mathcal{N}\mathcal{E}), \\ \nabla \cdot \mathcal{E} = \bar{n} - \mathcal{N}, \quad \nabla \times \mathcal{E} = 0, \\ \mathcal{N}(0, x) = n_0, \end{cases}$$

which is a system of diffusion equations for the electron density, and maintains the parabolic-elliptic character.

Lastly, we study the combined non-relativistic and relaxation limits in the Euler-Maxwell equations (1.1) (*i.e.* $\varepsilon, \tau \to 0$). Set $\lambda = 1$. From the "$\mathcal{O}(1/\tau)$ time scale" in (1.7), where the superscript τ is replaced by (τ, ε), the new variable $(n^{(\tau,\varepsilon)}, \mathbf{u}^{(\tau,\varepsilon)}, \mathbf{E}^{(\tau,\varepsilon)}, \mathbf{B}^{(\tau,\varepsilon)})$ satisfies

$$(1.10) \qquad \begin{cases} \partial_t n^{(\tau,\varepsilon)} + \nabla \cdot (n^{(\tau,\varepsilon)} \mathbf{u}^{(\tau,\varepsilon)}) = 0, \\ \tau^2 \partial_t(n^{(\tau,\varepsilon)} \mathbf{u}^{(\tau,\varepsilon)}) + \tau^2 \nabla \cdot (n^{(\tau,\varepsilon)} \mathbf{u}^{(\tau,\varepsilon)} \otimes \mathbf{u}^{(\tau,\varepsilon)}) + \nabla P(n^{(\tau,\varepsilon)}) \\ \qquad\qquad = -n^{(\tau,\varepsilon)}\left(\mathbf{E}^{(\tau,\varepsilon)} + \tau\varepsilon \mathbf{u}^{(\tau,\varepsilon)} \times \mathbf{B}^{(\tau,\varepsilon)}\right) - n^{(\tau,\varepsilon)} \mathbf{u}^{(\tau,\varepsilon)}, \\ \tau\varepsilon \partial_t \mathbf{E}^{(\tau,\varepsilon)} - \nabla \times \mathbf{B}^{(\tau,\varepsilon)} = \tau\varepsilon n^{(\tau,\varepsilon)} \mathbf{u}^{(\tau,\varepsilon)}, \\ \tau\varepsilon \partial_t \mathbf{B}^{(\tau,\varepsilon)} + \nabla \times \mathbf{E}^{(\tau,\varepsilon)} = 0, \\ \nabla \cdot \mathbf{E}^{(\tau,\varepsilon)} = \bar{n} - n^{(\tau,\varepsilon)}, \quad \nabla \cdot \mathbf{B}^{(\tau,\varepsilon)} = 0. \end{cases}$$

Obviously, in the process of combined limits $\tau, \varepsilon \to 0$, the limits $\mathcal{N}, \mathcal{E}$ of $n^{(\tau,\varepsilon)}, \mathbf{E}^{(\tau,\varepsilon)}$ also satisfy the drift-diffusion equations (1.9).

1.2. Main results. In the past ten years, the Euler-Poisson equations (1.6) have attracted much attention. There are many contributions in mathematical analysis, such as the well-posedness of steady-state solutions, global existence of classical or entropy weak solutions, large time behavior of classical solutions, relaxation limit problems and so on, the reader is referred to [2, 8, 11, 12, 13, 14, 15, 19, 21] and the references therein, also including ourselves [10, 27, 28], while the Euler-Maxwell equations are much more intricate than the Euler-Poisson equations, not only because of Maxwell's equations, but also because of the complicated coupling of the Lorentz force $(\mathbf{E} + \mathbf{u} \times \mathbf{B})$. In contrast, not so many works have been devoted to the study of Euler-Maxwell equations. Up to now, only partial results are available.

Using the Godunov scheme with the fractional step and the compensated compactness theory, Chen, Jerome and Wang [6] constructed the existence of a global weak solution to the initial boundary value problem for arbitrarily large initial data in $L^\infty(\mathbf{R})$. In [16], assuming initial data in Sobolev spaces $H^s(\mathbf{R}^3)$ with higher regularity ($s > 5/2$), a local existence theory of smooth solutions for the Cauchy problem of non-isentropic Euler-Maxwell equations, where the pressure-density function (1.2) is replaced with the energy equation, was established by modificating the classical semigroup-resolvent approach of Kato [17]. In [23, 24, 25], based on the existence theory of Kato and Majda [17, 20], Peng and Wang justified the non-relativistic limit ($\varepsilon \to 0$), the quasi-neutral limit ($\lambda \to 0$) and the combined non-relativistic and quasi-neutral limits ($\varepsilon = \lambda \to 0$) for the Euler-Maxwell equations (1.1) in virtue of the analysis of asymptotic expansions. Their results show that the Euler-Maxwell equations converge towards the Euler-Poisson equations, e-MHD system and incompressible Euler equations in some time-interval independent of the parameters ε and λ, respectively.

However, the well-posedness and singular limits for the Euler-Maxwell equations (1.1) in several dimensions are still far from well-known, in particular, in the framework of critical spaces. In the present paper, we shall answer this problem. More concretely speaking, we shall consider a small perturbation near the constant equilibrium state $(\bar{n}, \mathbf{0}, \mathbf{0}, \overline{\mathbf{B}})$ which is a particular solution of the Cauchy problem (1.1)-(1.3), and obtain the global existence and uniqueness of classical solutions. We choose the critical Besov spaces in space-variable x as the basic functional setting, where the regularity index ($\sigma = 1 + N/2$) is just the limit case of classical existence theory of Kato and Majda [17, 20]. Although this idea has been used to study the compressible Euler-Poisson equations (1.6) in [10, 27, 28] recently, it should be pointed out that the Euler-Maxwell equations are essentially different from (1.6). In comparison with the methods in [10, 27, 28], we have to face with several technical difficulties arising in the uniform *a priori* estimates of classical solutions in critical spaces. The first one is lack of the low-frequency estimate of magnetic field $\mathbf{B}$, which does not lead to the exponential decay near equilibrium in view of the standard definition of norm of Besov spaces. Another one is that the nonlinear terms (pressure, Lorentz field, etc.) will hinder us establishing the uniform estimates with respect to the singular parameter couple (τ, ε). To overcome these difficulties, we add the new content in the proof of the local existence and (uniform) global existence of classical solutions. Actually, the Chemin-Lerner's spaces $\widetilde{L}_T^\rho(B_{p,r}^s)$ in [4] are introduced, which is a refinement of the usual spaces $L_T^\rho(B_{p,r}^s)$, and some uniform frequency-localization estimates in Chemin-Lerner's spaces with critical regularity are established, for details, see Lemmas 3.3-3.4 and Lemmas 3.6-3.7. Based on the uniform estimates, we further rigorously justify the singular limit problems for (1.1)-(1.3) in Sect. 1.1 by the standard weak convergence methods and the application of compactness theorem in [26].

Throughout this paper, the regularity index $\sigma = 1 + N/2$. First of all, we state a local existence and uniqueness theorem of classical solutions to the Cauchy problem (1.1)-(1.3) away from the vacuum.

THEOREM 1.1. *Let $\bar{n} > 0$ be a constant reference density and $\overline{\mathbf{B}} \in \mathbf{R}^N$ be any given constant. Suppose that $n_0 - \bar{n}, \mathbf{u}_0, \mathbf{E}_0$ and $\mathbf{B}_0 - \overline{\mathbf{B}} \in B_{2,1}^\sigma(\mathbf{R}^N)$ satisfy $n_0 > 0$ and the compatible conditions (1.4), then there exist a time $T_0 > 0$ and a unique solution $(n, \mathbf{u}, \mathbf{E}, \mathbf{B})$ of the system (1.1)-(1.3) such that*

$$(n, \mathbf{u}, \mathbf{E}, \mathbf{B}) \in \mathcal{C}^1([0, T_0] \times \mathbf{R}^N) \quad with \quad n > 0 \ for \ all \ t \in [0, T_0]$$

and

$$(n - \bar{n}, \mathbf{u}, \mathbf{E}, \mathbf{B} - \overline{\mathbf{B}}) \in \widetilde{\mathcal{C}}_{T_0}(B_{2,1}^{\sigma}) \cap \widetilde{\mathcal{C}}_{T_0}^1(B_{2,1}^{\sigma-1}).$$

Remark 1.1. To avoid excessive commutators arising from the nonlinear pressure term by using Fourier frequency-localization method, we introduce a function transform in Sect. 3.1 such that the Euler-Maxwell equations (1.1) is reduced to a symmetric hyperbolic system. Based on the previous effort in [10], we obtain the local existence of classical solutions in the Chemin-Lerner's space with critical regularity (Proposition 3.1). Theorem 1.1 follows from Proposition 3.1 and Remark 3.1 readily. As a matter of fact, the new result is applicable to generally symmetrizable hyperbolic systems, which enriches and develops the classical existence theory of Kato and Majda [17, 20].

In small amplitude regime, we get the uniform global well-posedness of classical solutions to the Cauchy problem (1.1)-(1.3) in critical spaces. From now on, we set the scaled Debye length to be one ($\lambda \equiv 1$).

THEOREM 1.2. *Let $\bar{n} > 0$ be a constant reference density and $\overline{\mathbf{B}} \in \mathbf{R}^N$ be any given constant. Suppose that $n_0 - \bar{n}, \mathbf{u}_0, \mathbf{E}_0$ and $\mathbf{B}_0 - \overline{\mathbf{B}} \in B_{2,1}^{\sigma}(\mathbf{R}^N)$ satisfy the compatible conditions (1.4). There exists a positive constant δ_0 independent of singular parameter couple (τ, ε), such that if*

$$\|(n_0 - \bar{n}, \mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0 - \overline{\mathbf{B}})\|_{B_{2,1}^{\sigma}} \leq \delta_0,$$

then there exists a unique global solution $(n, \mathbf{u}, \mathbf{E}, \mathbf{B})$ of the system (1.1)-(1.3) satisfying

$$(n, \mathbf{u}, \mathbf{E}, \mathbf{B}) \in \mathcal{C}^1([0, \infty) \times \mathbf{R}^N)$$

and

$$(n - \bar{n}, \mathbf{u}, \mathbf{E}, \mathbf{B} - \overline{\mathbf{B}}) \in \widetilde{\mathcal{C}}(B_{2,1}^{\sigma}(\mathbf{R}^N)) \cap \widetilde{\mathcal{C}}^1(B_{2,1}^{\sigma-1}(\mathbf{R}^N)).$$

Moreover, the uniform energy estimate holds:

$$\|(n - \bar{n}, \mathbf{u}, \mathbf{E}, \mathbf{B} - \overline{\mathbf{B}})\|_{\widetilde{L}^{\infty}(B_{2,1}^{\sigma})}$$

$$+ \mu_0 \Big\{ \Big\| \Big(\sqrt{\tau}(n - \bar{n}), \frac{1}{\sqrt{\tau}}\mathbf{u}, \sqrt{\tau\varepsilon}\mathbf{E}\Big) \Big\|_{\widetilde{L}^2(B_{2,1}^{\sigma})} + \Big\| \frac{1}{\sqrt{\varepsilon}}\nabla\mathbf{B} \Big\|_{\widetilde{L}^2(B_{2,1}^{\sigma-1})} \Big\}$$

$$(1.11) \quad \leq C_0 \|(n_0 - \bar{n}, \mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0 - \overline{\mathbf{B}})\|_{B_{2,1}^{\sigma}}$$

for $0 < \tau, \varepsilon \leq 1$, where the positive constants μ_0, C_0 are independent of (τ, ε).

Remark 1.2. Together with Theorem 1.1, Theorem 1.2 directly follows from the standard continuation argument and the crucial energy estimate (1.11) which presents the dissipation rates of all the components in the solution. Noticing that the coupled electromagnetic field $(\mathbf{E}, \mathbf{B})$ appears in the nonlinear source terms of Euler system, which indeed does not affect the character of corresponding linearized form, so we can take the full advantage of "Shizuta-Kawashima" skew-symmetry condition which was well developed for general hyperbolic systems of balance laws [18, 29] to capture the dissipation rate of density function, see Lemma 3.4. On the other hand, from the proof of Lemmas 3.6-3.7, we see that the electromagnetic field generated by the

compressible electron flow exhibits a weak dissipation property, which is essentially different from the pure Maxwell's equations, although the low-frequency estimate of dissipation rate of $\mathbf{B}$ is absent. In addition, we track the singular parameters τ and ε in the proof of (1.11), which plays a key role in the study of related limit problems.

As a direct consequence of Theorem 1.2, we can obtain the large-time asymptotic behavior of global solutions near the equilibrium $(\bar{n}, \mathbf{0}, \mathbf{0}, \overline{\mathbf{B}})$ in some Besov spaces.

COROLLARY 1.3. *Let $(n, \mathbf{u}, \mathbf{E}, \mathbf{B})$ be the solution in Theorem 1.2, it holds that*

$$\|n(\cdot, t) - \bar{n}, \mathbf{u}(\cdot, t), \mathbf{E}(\cdot, t)\|_{B_{2,1}^{\sigma-\varepsilon'}(\mathbf{R}^N)} \to 0, \qquad \|\mathbf{B}(\cdot, t) - \overline{\mathbf{B}}\|_{B_{p,1}^{\sigma-1-\varepsilon'}(\mathbf{R}^N)} \to 0,$$

as the time variable $t \to +\infty$, where $p = \frac{2N}{N-2}(N > 2)$ and $\varepsilon' > 0$.

Remark 1.3. Recalling the proof of Corollary 5.1 in [9], Corollary 1.3 is followed by a minor revision. The definite convergence rate to the equilibrium $(\bar{n}, \mathbf{0}, \mathbf{0}, \overline{\mathbf{B}})$ will be studied in the future work.

Next, we state the non-relativistic limit of uniform global solutions to (1.1)-(1.3) for any fixed momentum relaxation time $\tau > 0$.

THEOREM 1.4 (**Non-relativistic limit**). *Let $\tau = 1$ and $(n^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{E}^\varepsilon, \mathbf{B}^\varepsilon)$ be the global solution of (1.1)-(1.3) given by Theorem 1.2. Then there exists some function $(n^0, \mathbf{u}^0, \mathbf{E}^0)$ which is a global solution to the Euler-Poisson equations (1.6) satisfying $(n^0 - \bar{n}, \mathbf{u}^0, \mathbf{E}^0) \in \mathcal{C}([0, \infty), B_{2,1}^\sigma(\mathbf{R}^N))$ such that as $\varepsilon \to 0$, it holds that*

$$(n^\varepsilon, \mathbf{u}^\varepsilon, \sqrt{\varepsilon}\mathbf{E}^\varepsilon) \to (n^0, \mathbf{u}^0, \mathbf{0}) \quad \textit{strongly in} \quad \mathcal{C}([0, T], (B_{2,1}^{\sigma-\delta}(\mathbf{R}^N))_{\text{loc}}),$$

$$\nabla \mathbf{B}^\varepsilon \to \mathbf{0} \quad \textit{strongly in} \quad L_T^2(B_{2,1}^{\sigma-1}(\mathbf{R}^N)),$$

$$(\mathbf{E}^\varepsilon, \mathbf{B}^\varepsilon) \rightharpoonup (\mathbf{E}^0, \overline{\mathbf{B}}) \quad \textit{weakly}^\star \textit{ in} \quad L_T^\infty(B_{2,1}^\sigma(\mathbf{R}^N)),$$

for any $T > 0$ and $\delta \in (0, 1)$. Moreover, it yields

$$\|(n^0 - \bar{n}, \mathbf{u}^0, \mathbf{E}^0)(t, \cdot)\|_{B_{2,1}^\sigma(\mathbf{R}^N)}$$
$$(1.12) \qquad \leq C_1 \|(n_0 - \bar{n}, \mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0 - \overline{\mathbf{B}})\|_{B_{2,1}^\sigma(\mathbf{R}^N)}, \ t \geq 0,$$

where $C_1 > 0$ is a uniform constant independent of ε.

Secondly, we justify the relaxation limit for the Euler-Maxwell equations (1.1). To this end, we consider the Cauchy problem for the re-scaled system (1.8) subject to the initial data

$$(1.13) \qquad (n^\tau, \mathbf{u}^\tau, \mathbf{E}^\tau, \mathbf{B}^\tau)(0, x) = \left(n_0, \frac{1}{\tau}\mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0\right)(x).$$

It follows from Theorem 1.2 and the "$\mathcal{O}(1/\tau)$ time scale" (1.7) that there exists a unique global in-time classical solution $(n^\tau, \mathbf{u}^\tau, \mathbf{E}^\tau, \mathbf{B}^\tau)$ to the system (1.8) and (1.13). Then, we have

THEOREM 1.5 (**Relaxation limit**). *Let $\varepsilon = 1$ and $(n^\tau, \mathbf{u}^\tau, \mathbf{E}^\tau, \mathbf{B}^\tau)$ be the global solution of (1.8) and (1.13) obtained from Theorem 1.2. Then, there exists a function $(\mathcal{N}, \mathcal{U}, \mathcal{E})$ which is a global solution to the drift-diffusion equations (1.9) satisfying*

$$(\mathcal{N}, \mathcal{U}, \mathcal{E}) \in \mathcal{C}([0, \infty), B_{2,1}^\sigma(\mathbf{R}^N)) \times L^2([0, \infty), B_{2,1}^\sigma(\mathbf{R}^N)) \times \mathcal{C}([0, \infty), B_{2,1}^\sigma(\mathbf{R}^N))$$

such that as $\tau \to 0$, it holds that

$$(n^\tau, \tau^2 \mathbf{u}^\tau, \sqrt{\tau} \mathbf{E}^\tau) \to (\mathcal{N}, \mathbf{0}, \mathbf{0}) \quad \text{strongly in} \quad \mathcal{C}([0, T], (B_{2,1}^{\sigma-\delta}(\mathbf{R}^N))_{\mathrm{loc}}),$$

$$\mathbf{u}^\tau \rightharpoonup \mathcal{U} \quad \text{weakly in} \quad L_T^2(B_{2,1}^\sigma(\mathbf{R}^N)),$$

$$\nabla \mathbf{B}^\tau \to \mathbf{0} \quad \text{strongly in} \quad L_T^2(B_{2,1}^{\sigma-1}(\mathbf{R}^N)),$$

$$(\mathbf{E}^\tau, \mathbf{B}^\tau) \rightharpoonup (\mathcal{E}, \overline{\mathbf{B}}) \quad \text{weakly}^\star \text{ in} \quad L_T^\infty(B_{2,1}^\sigma(\mathbf{R}^N)),$$

for any $T > 0$ and $\delta \in (0, 1)$. Moreover, it yields

(1.14)
$$\|(\mathcal{N} - \bar{n}, \mathcal{E})(t, \cdot)\|_{B_{2,1}^\sigma(\mathbf{R}^N)}$$
$$\leq C_2 \|(n_0 - \bar{n}, \mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0 - \overline{\mathbf{B}})\|_{B_{2,1}^\sigma(\mathbf{R}^N)}, \ t \geq 0,$$

where $C_2 > 0$ is a uniform constant independent of τ.

Finally, what left is the combined non-relativistic and relaxation limits for (1.1). From Theorem 1.2 and (1.13) where the superscript τ is replaced by (τ, ε), it is shown that there exists a unique global in-time classical solution $(n^{(\tau,\varepsilon)}, \mathbf{u}^{(\tau,\varepsilon)}, \mathbf{E}^{(\tau,\varepsilon)}, \mathbf{B}^{(\tau,\varepsilon)})$ to the system (1.10) and (1.13). Furthermore, we get

THEOREM 1.6 (**Combined non-relativistic and relaxation limits**). *Let $(n^{(\tau,\varepsilon)}, \mathbf{u}^{(\tau,\varepsilon)}, \mathbf{E}^{(\tau,\varepsilon)}, \mathbf{B}^{(\tau,\varepsilon)})$ be the global solution of (1.10) and (1.13) obtained from Theorem 1.2. Then, there exists a function $(\mathcal{N}, \mathcal{U}, \mathcal{E})$ which is a global solution to the drift-diffusion equations (1.9) satisfying*

$$(\mathcal{N}, \mathcal{U}, \mathcal{E}) \in \mathcal{C}([0, \infty), B_{2,1}^\sigma(\mathbf{R}^N)) \times L^2([0, \infty), B_{2,1}^\sigma(\mathbf{R}^N)) \times \mathcal{C}([0, \infty), B_{2,1}^\sigma(\mathbf{R}^N))$$

such that as $\tau \to 0$ and $\varepsilon \to 0$ simultaneously, it holds that

$$(n^{(\tau,\varepsilon)}, \tau^2 \mathbf{u}^{(\tau,\varepsilon)}, \sqrt{\tau\varepsilon} \mathbf{E}^{(\tau,\varepsilon)}) \to (\mathcal{N}, \mathbf{0}, \mathbf{0}) \quad \text{strongly in} \quad \mathcal{C}([0, T], (B_{2,1}^{\sigma-\delta}(\mathbf{R}^N))_{\mathrm{loc}}),$$

$$\mathbf{u}^{(\tau,\varepsilon)} \rightharpoonup \mathcal{U} \quad \text{weakly in} \quad L_T^2(B_{2,1}^\sigma(\mathbf{R}^N)),$$

$$\nabla \mathbf{B}^{(\tau,\varepsilon)} \to \mathbf{0} \quad \text{strongly in} \quad L_T^2(B_{2,1}^{\sigma-1}(\mathbf{R}^N)),$$

$$(\mathbf{E}^{(\tau,\varepsilon)}, \mathbf{B}^{(\tau,\varepsilon)}) \rightharpoonup (\mathcal{E}, \overline{\mathbf{B}}) \quad \text{weakly}^\star \text{ in} \quad L_T^\infty(B_{2,1}^\sigma(\mathbf{R}^N)),$$

for any $T > 0$ and $\delta \in (0, 1)$. Moreover, it yields

(1.15)
$$\|(\mathcal{N} - \bar{n}, \mathcal{E})(t, \cdot)\|_{B_{2,1}^\sigma(\mathbf{R}^N)}$$
$$\leq C_3 \|(n_0 - \bar{n}, \mathbf{u}_0, \mathbf{E}_0, \mathbf{B}_0 - \overline{\mathbf{B}})\|_{B_{2,1}^\sigma(\mathbf{R}^N)}, \ t \geq 0,$$

where $C_3 > 0$ is a uniform constant independent of (τ, ε).

Remark 1.4. To the best of our knowledge, these limit results (Theorems 1.4-1.6) show the convergence *globally* in time, which have not been appeared in the published literatures. In comparison with that in [23, 24, 25], they hold true in the functional

spaces with relatively *lower* regularity, which can be regarded as a supplement to the theory of singular limits for the Euler-Maxwell equations (1.1). Let us mention that the combined non-relativistic and relaxation limits obtained in Theorem 1.6 does not require any (communication) restriction between τ and ε. That is, one can fix any of the two parameters τ and ε and let the other tends to zero, which is the *genuinely* combined limits.

Remark 1.5. It is worth noting that Chemin-Lerner's spaces are first introduced to establish the uniform *a priori* estimates with respect to (τ, ε) and justify the combined limits. As a matter fact, this approach developed by the current paper can be applied to study other limit problems with two (or more) independent singular parameters.

Remark 1.6. There is no additional conceptual difficulty in considering the temperature effects and the corresponding balance equation (*i.e.* non-isentropic Euler-Maxwell equations), although the estimates are quite tedious.

The rest of this paper unfolds as follows. In Sect. 2, we introduce the Littlewood-Paley decomposition and recall the definitions and some useful results on Besov spaces and Chemin-Lerner's spaces. Sect. 3 is devoted to the proofs of main results, which is divided into five subsections for clarity. In Sect. 3.1, we first rewrite the Euler-Maxwell equations (1.1) as a symmetric hyperbolic system in order to obtain the effective *a priori* estimate by using Fourier frequency localization. Furthermore, we give the local existence of classical solutions in Chemin-Lerner's spaces with critical regularity. Then in Sect. 3.2, we deduce a new uniform *a priori* estimate under some smallness assumption, which is used to achieve the (uniform) global existence of classical solutions. Sect. 3.3, Sect. 3.4 and Sect. 3.5 are in turn dedicated to the justification of the non-relativistic limit, relaxation limit as well as combined non-relativistic and relaxation limits of Euler-Maxwell equations.

Notations. Throughout the paper, C stands for a uniform positive constant with respect to (τ, ε). The notation $f \approx g$ means that $f \leq Cg$ and $g \leq Cf$. Denote by $\mathcal{C}([0,T], X)$ (resp., $\mathcal{C}^1([0,T], X)$) the space of continuous (resp., continuously differentiable) functions on $[0,T]$ with values in a Banach space X. For simplicity, the notation $\|(f, g, h, k)\|_X$ means $\|f\|_X + \|g\|_X + \|h\|_X + \|k\|_X$, where $f, g, h, k \in X$. We omit the space dependence, since all functional spaces (in x) are considered in $\mathbf{R}^N$. Moreover, the integral $\int_{\mathbf{R}^N} f dx$ is labeled as $\int f$ without any ambiguity.

2. Tools. The proofs of most of the results presented in this paper require a dyadic decomposition of Fourier variable. Let us recall briefly the Littlewood-Paley decomposition theory and the characterization of Besov spaces and Chemin-Lerner's spaces, see for instance [3, 7] for details.

Let (φ, χ) be a couple of smooth functions valued in $[0, 1]$ such that φ is supported in the shell $\mathcal{C}(0, \frac{3}{4}, \frac{8}{3}) = \{\xi \in \mathbf{R}^N | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, χ is supported in the ball $\mathcal{B}(0, \frac{4}{3}) = \{\xi \in \mathbf{R}^N | |\xi| \leq \frac{4}{3}\}$ and

$$\chi(\xi) + \sum_{q=0}^{\infty} \varphi(2^{-q}\xi) = 1, \quad q \in \mathbf{Z}, \ \xi \in \mathbf{R}^N.$$

Let $\mathcal{S}'$ be the dual space of the Schwartz class $\mathcal{S}$. For $f \in \mathcal{S}'$, the nonhomogeneous dyadic blocks are defined as follows

$$\Delta_{-1} f := \chi(D) f = \tilde{h} * f \quad \text{with} \ \tilde{h} = \mathcal{F}^{-1}\chi,$$

$$\Delta_q f := \varphi(2^{-q}D) f = 2^{qd} \int h(2^q y) f(x - y) dy \quad \text{with} \ h = \mathcal{F}^{-1}\varphi, \ \text{if} \ q \geq 0.$$

Here $*$, $\mathcal{F}^{-1}$ represent the convolution operator and the inverse Fourier transform, respectively. Note that $\tilde{h}, h \in \mathcal{S}$. The nonhomogeneous Littlewood-Paley decomposition is

$$f = \sum_{q \geq -1} \Delta_q f \quad \text{in} \quad \mathcal{S}'.$$

Define the low frequency cut-off by

$$S_q f := \sum_{p \leq q-1} \Delta_p f.$$

According to the above Littlewood-Paley decomposition, thus we introduce the explicit definition of Besov spaces.

DEFINITION 2.1. *Let $1 \leq p \leq \infty$ and $s \in \mathbf{R}$. For $1 \leq r < \infty$, the Besov spaces $B_{p,r}^s$ are defined by*

$$f \in B_{p,r}^s \Leftrightarrow \Big(\sum_{q \geq -1} (2^{qs} \|\Delta_q f\|_{L^p})^r \Big)^{\frac{1}{r}} < \infty$$

and $B_{p,\infty}^s$ are defined by

$$f \in B_{p,\infty}^s \Leftrightarrow \sup_{q \geq -1} 2^{qs} \|\Delta_q f\|_{L^p} < \infty.$$

Let us point out that the definition of $B_{p,r}^s$ does not depend on the choice of the Littlewood-Paley decomposition. Now, we state some classical conclusions, which will be used in subsequent analysis. The first one is Bernstein's inequality.

LEMMA 2.2. *Let $k \in \mathbf{N}$ and $0 < R_1 < R_2$. There exists a constant C, depending only on R_1, R_2 and N, such that for all $1 \leq a \leq b \leq \infty$ and $f \in L^a$, we have*

$$\text{Supp } \mathcal{F} f \subset \mathcal{B}(0, R_1 \lambda) \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+N(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a};$$

$$\text{Supp } \mathcal{F} f \subset \mathcal{C}(0, R_1 \lambda, R_2 \lambda) \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C^{k+1} \lambda^k \|f\|_{L^a},$$

where $\mathcal{F} f$ represents the Fourier transform on f.

The second one is a compactness result for Besov spaces.

PROPOSITION 2.3. *Let $1 \leq p, r \leq \infty$, $s \in \mathbf{R}$ and $\varepsilon > 0$. For all $\phi \in C_c^\infty$, the map $f \mapsto \phi f$ is compact from $B_{p,r}^{s+\varepsilon}$ to $B_{p,r}^s$.*

On the other hand, the study of non-stationary partial differential equations requires spaces of type $L_T^\rho(X) := L^\rho(0, T; X)$ for appropriate Banach spaces X. In our case, X is expected to be a Besov space, so the fundamental idea is to localize the equations through the Littlewood-Paley decomposition. Then it is easy to obtain $L_T^\rho(L^p)$ estimates for each dyadic block. Performing a (weighted) ℓ^r summation is the most natural next step. But, in doing so, we get bounds in spaces which are not type $L_T^\rho(B_{p,r}^s)$ (except if $\rho = r$). This leads to the definition of Chemin-Lerner's spaces first introduced by J.-Y. Chemin and N. Lerner [4], which is the refinement of the spaces $L_T^\rho(B_{p,r}^s)$.

DEFINITION 2.4. *For $T > 0, s \in \mathbf{R}, 1 \leq r, \rho \leq \infty$, set (with the usual convention if $r = \infty$)*

$$\|f\|_{\widetilde{L}_T^\rho(B_{p,r}^s)} := \Big(\sum_{q \geq -1} (2^{qs} \|\Delta_q f\|_{L_T^\rho(L^p)})^r \Big)^{\frac{1}{r}}.$$

Then we define the space $\widetilde{L}_T^\rho(B_{p,r}^s)$ *as the completion of* $\mathcal{S}$ *over* $(0,T) \times \mathbf{R}^N$ *by the above norm.*

Furthermore, we define

$$\widetilde{\mathcal{C}}_T(B_{p,r}^s) := \widetilde{L}_T^\infty(B_{p,r}^s) \cap \mathcal{C}([0,T], B_{p,r}^s)$$

and

$$\widetilde{\mathcal{C}}_T^1(B_{p,r}^s) := \{f \in \mathcal{C}^1([0,T], B_{p,r}^s) | \partial_t f \in \widetilde{L}_T^\infty(B_{p,r}^s)\}.$$

The index T will be omitted when $T = +\infty$. Let us emphasize that

Remark 2.1. According to Minkowski's inequality, it holds that

$$\|f\|_{\widetilde{L}_T^\rho(B_{p,r}^s)} \leq \|f\|_{L_T^\rho(B_{p,r}^s)} \text{ if } r \geq \rho, \qquad \|f\|_{\widetilde{L}_T^\rho(B_{p,r}^s)} \geq \|f\|_{L_T^\rho(B_{p,r}^s)} \text{ if } r \leq \rho.$$

Then, we state the property of continuity for product in Chemin-Lerner's spaces $\widetilde{L}_T^\rho(B_{p,r}^s)$.

PROPOSITION 2.5. *The following estimate holds:*

$$\|fg\|_{\widetilde{L}_T^\rho(B_{p,r}^s)} \leq C(\|f\|_{L_T^{\rho_1}(L^\infty)} \|g\|_{\widetilde{L}_T^{\rho_2}(B_{p,r}^s)} + \|g\|_{L_T^{\rho_3}(L^\infty)} \|f\|_{\widetilde{L}_T^{\rho_4}(B_{p,r}^s)})$$

whenever $s > 0, 1 \leq p \leq \infty, 1 \leq \rho, \rho_1, \rho_2, \rho_3, \rho_4 \leq \infty$ *and*

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho_3} + \frac{1}{\rho_4}.$$

As a direct corollary, one has

$$\|fg\|_{\widetilde{L}_T^\rho(B_{p,r}^s)} \leq C \|f\|_{\widetilde{L}_T^{\rho_1}(B_{p,r}^s)} \|g\|_{\widetilde{L}_T^{\rho_2}(B_{p,r}^s)}$$

whenever $s \geq N/p, \frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$.

In addition, the estimates of commutators in $\widetilde{L}_T^\rho(B_{p,1}^s)$ spaces are also frequently used in the subsequent analysis. The indices s, p behave just as in the stationary case [7, 10] whereas the time exponent ρ behaves according to Hölder inequality.

LEMMA 2.6. *Let* $1 < p < \infty$ *and* $1 \leq \rho \leq \infty$*, then the following inequalities are true:*

$$2^{qs} \|[f, \Delta_q]\mathcal{A}g\|_{L_T^\rho(L^p)}$$

$$\leq \begin{cases} Cc_q \|f\|_{\widetilde{L}_T^{\rho_1}(B_{p,1}^s)} \|g\|_{\widetilde{L}_T^{\rho_2}(B_{p,1}^s)}, & s = 1 + N/p, \\ Cc_q \|f\|_{\widetilde{L}_T^{\rho_1}(B_{p,1}^s)} \|g\|_{\widetilde{L}_T^{\rho_2}(B_{p,1}^{s+1})}, & s = N/p, \\ Cc_q \|f\|_{\widetilde{L}_T^{\rho_1}(B_{p,1}^{s+1})} \|g\|_{\widetilde{L}_T^{\rho_2}(B_{p,1}^s)}, & s = N/p, \end{cases}$$

where the commutator $[\cdot, \cdot]$ *is defined by* $[f,g] = fg - gf$*, the operator* $\mathcal{A} = \operatorname{div} \text{ or } \nabla$*, C is a harmless constant, and* c_q *denotes a sequence such that* $\|(c_q)\|_{l^1} \leq 1, \frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$.

Finally, we state a continuity result for compositions (see [1]) to end up this section.

PROPOSITION 2.7. *Let* $s > 0$*,* $1 \leq p, r, \rho \leq \infty$*,* $F \in W_{loc}^{[s]+1,\infty}(I; \mathbf{R})$ *with* $F(0) = 0$*,* $T \in (0, \infty]$ *and* $v \in \widetilde{L}_T^\rho(B_{p,r}^s) \cap L_T^\infty(L^\infty)$*. Then*

$$\|F(v)\|_{\widetilde{L}_T^\rho(B_{p,r}^s)} \leq C(1 + \|v\|_{L_T^\infty(L^\infty)})^{[s]+1} \|v\|_{\widetilde{L}_T^\rho(B_{p,r}^s)}.$$

3. The proofs of main results. In what follows, we focus on the proofs of main results. For clarity, we divide them into several subsections, since the proofs are a bit longer.

3.1. Reformulation and local existence. In this section, we reformulate (1.1)-(1.3) in order to obtain the effective *a priori* estimates by means of Fourier frequency localization.

For the isentropic case ($\gamma > 1$), let us introduce the sound speed

$$\psi(n) = \sqrt{P'(n)},$$

and set $\bar{\psi} = \psi(\bar{n})$ corresponding to the sound speed at a background density $\bar{n}$. Define

(3.1) $$\varrho = \frac{2}{\gamma - 1}\Big(\psi(n) - \bar{\psi}\Big), \quad \mathbf{F} = \mathbf{B} - \overline{\mathbf{B}}.$$

Set

$$W := (\varrho, \mathbf{u}, \mathbf{E}, \mathbf{F})^{\top}.$$

Then the system (1.1) can be reduced to the symmetric hyperbolic system for smooth solutions:

(3.2) $$\begin{cases} \partial_t \varrho + \bar{\psi}\mathrm{div}\mathbf{u} = -\mathbf{u} \cdot \nabla\varrho - \frac{\gamma-1}{2}\varrho\mathrm{div}\mathbf{u}, \\ \partial_t \mathbf{u} + \bar{\psi}\nabla\varrho + \frac{\mathbf{u}}{\tau} = -\mathbf{u} \cdot \nabla\mathbf{u} - \frac{\gamma-1}{2}\varrho\nabla\varrho - (\mathbf{E} + \varepsilon\mathbf{u} \times (\mathbf{F} + \overline{\mathbf{B}})), \\ \partial_t \mathbf{E} - \frac{1}{\varepsilon}\nabla \times \mathbf{F} = \bar{n}\mathbf{u} + h(\varrho)\mathbf{u}, \\ \partial_t \mathbf{F} + \frac{1}{\varepsilon}\nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{E} = -h(\varrho), \quad \nabla \cdot \mathbf{F} = 0, \end{cases}$$

where $h(\varrho) = \{(P_0\gamma)^{-\frac{1}{2}}(\frac{\gamma-1}{2}\varrho + \bar{\psi})\}^{\frac{2}{\gamma-1}} - \bar{n}$ is a smooth function on the domain $\{\varrho|\frac{\gamma-1}{2}\varrho + \bar{\psi} > 0\}$ satisfying $h(0) = 0$. The initial data (1.3) becomes into

(3.3) $$W|_{t=0} = (\varrho_0, \mathbf{u}_0, \mathbf{E}_0, \mathbf{F}_0)$$

with

$$\varrho_0 = \frac{2}{\gamma - 1}\Big(\psi(n_0) - \bar{\psi}\Big), \quad \mathbf{F}_0 = \mathbf{B}_0 - \overline{\mathbf{B}}.$$

Under the symmetrization transform (3.1), the initial data (3.3) satisfies the corresponding compatible conditions

(3.4) $$\nabla \cdot \mathbf{E}_0 = -h(\varrho_0), \quad \nabla \cdot \mathbf{F_0} = 0, \quad x \in \mathbf{R}^N.$$

Remark 3.1. The variable change is from the open set $\{(n, \mathbf{u}, \mathbf{E}, \mathbf{B}) \in (0, +\infty) \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N\}$ to the open set $\{W \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N|\frac{\gamma-1}{2}\varrho + \bar{\psi} > 0\}$. It is easy to show that for classical solutions $(n, \mathbf{u}, \mathbf{E}, \mathbf{B})$ away from vacuum, (1.1)-(1.3) is equivalent to (3.2)-(3.3) with $\frac{\gamma-1}{2}\varrho + \bar{\psi} > 0$.

For the isothermal case where $\gamma = 1$, the form of (3.2) is still valid with $\bar{\psi} = \sqrt{P_0}$, while the symmetrization transform depends on the following enthalpy variable change

(3.5) $$\varrho = \sqrt{P_0}(\ln n - \ln \bar{n}),$$

for the details, see e.g. [10].

Without loss of generality, we shall study the system (3.2)-(3.3) for $\gamma > 1$ and prove main results, since the case of $\gamma = 1$ can be discussed in the same way.

In [17, 20], Kato and Majda established a local existence theory for generally symmetric hyperbolic systems pertaining to data in the Sobolev spaces with higher regularity. Recently, using the regularized means and compactness argument, we have established a local existence in the framework critical Besov spaces for the Euler-Poisson equations (1.6), see [10]. In the present paper, we further strengthen the result such that it holds in Chemin-Lerner's spaces with critical regularity. Our result reads as follows.

PROPOSITION 3.1. *For any fixed $0 < \tau, \varepsilon \leq 1$, assume that $W_0 \in B_{2,1}^\sigma$ satisfying $\frac{\gamma-1}{2}\varrho_0 + \bar{\psi} > 0$ and (3.4), then there exist a time $T_0 > 0$ (depending only on the initial data) and a unique solution W to (3.2)-(3.3) such that $W \in \mathcal{C}^1([0, T_0] \times \mathbf{R}^N)$ with $\frac{\gamma-1}{2}\varrho + \bar{\psi} > 0$ for all $t \in [0, T_0]$ and $W \in \widetilde{\mathcal{C}}_{T_0}(B_{2,1}^\sigma) \cap \widetilde{\mathcal{C}}_{T_0}^1(B_{2,1}^{\sigma-1})$.*

Proof. Let the assumptions of Proposition 3.1 be fulfilled. By a proper revision, the local result in [10] can also be adapted to the Euler-Maxwell equations (3.2)-(3.3). That is, there exist a time $T_0 > 0$ (depending only on the initial data) and a unique solution W to (3.2)-(3.3) such that $W \in \mathcal{C}^1([0, T_0] \times \mathbf{R}^N)$ with $\frac{\gamma-1}{2}\varrho + \bar{\psi} > 0$ for all $t \in [0, T_0]$ and $W \in \mathcal{C}([0, T_0], B_{2,1}^\sigma) \cap \mathcal{C}^1([0, T_0], B_{2,1}^{\sigma-1})$. In order to prove Proposition 3.1, it suffices to show that $W \in \widetilde{L}_{T_0}^\infty(B_{2,1}^\sigma)$ and $W_t \in \widetilde{L}_{T_0}^\infty(B_{2,1}^{\sigma-1})$.

Indeed, applying the operator $\Delta_q(q \geq -1)$ to (3.2), we infer that $(\Delta_q\varrho, \Delta_q\mathbf{u}, \Delta_q\mathbf{E}, \Delta_q\mathbf{F})$ satisfies

$$(3.6) \quad \begin{cases} \partial_t\Delta_q\varrho + \bar{\psi}\Delta_q\mathrm{div}\mathbf{u} = -(\mathbf{u}\cdot\nabla)\Delta_q\varrho + [\mathbf{u}, \Delta_q]\cdot\nabla\varrho - \frac{\gamma-1}{2}\Delta_q(\varrho\mathrm{div}\mathbf{u}), \\ \partial_t\Delta_q\mathbf{u} + \bar{\psi}\Delta_q\nabla\varrho + \frac{1}{\tau}\Delta_q\mathbf{u} = -(\mathbf{u}\cdot\nabla)\Delta_q\mathbf{u} + [\mathbf{u}, \Delta_q]\cdot\nabla\mathbf{u} \\ \qquad\qquad - \frac{\gamma-1}{2}\Delta_q(\varrho\nabla\varrho) - \Delta_q\mathbf{E} - \varepsilon\Delta_q\mathbf{u}\times\overline{\mathbf{B}} - \varepsilon\Delta_q(\mathbf{u}\times\mathbf{F}), \\ \partial_t\Delta_q\mathbf{E} - \frac{1}{\varepsilon}\nabla\times\Delta_q\mathbf{F} = \bar{n}\Delta_q\mathbf{u} + \Delta_q(h(\varrho)\mathbf{u}), \\ \partial_t\Delta_q\mathbf{F} + \frac{1}{\varepsilon}\nabla\times\Delta_q\mathbf{E} = 0, \end{cases}$$

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] = fg - gf$.

Then multiplying the first equation of Eq. (3.6) by $\Delta_q\varrho$, the second one by $\Delta_q\mathbf{u}$, and adding the resulting equations together, after integrating it over $\mathbf{R}^N$, we have the energy equality

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta_q\varrho\|_{L^2}^2 + \|\Delta_q\mathbf{u}\|_{L^2}^2\right) + \frac{1}{\tau}\|\Delta_q\mathbf{u}\|_{L^2}^2$$

$$= \frac{1}{2}\int\mathrm{div}\mathbf{u}(|\Delta_q\varrho|^2 + |\Delta_q\mathbf{u}|^2) + \int([\mathbf{u}, \Delta_q]\cdot\nabla\varrho\Delta_q\varrho + [\mathbf{u}, \Delta_q]\cdot\nabla\mathbf{u}\Delta_q\mathbf{u})$$

$$+ \frac{\gamma-1}{2}\int\Delta_q\varrho(\nabla\varrho\cdot\Delta_q\mathbf{u}) - \frac{\gamma-1}{2}\int\left([\Delta_q, \varrho]\mathrm{div}\mathbf{u}\Delta_q\varrho + [\Delta_q, \varrho]\nabla\varrho\cdot\Delta_q\mathbf{u}\right)$$

$$(3.7) \quad - \int\Delta_q\mathbf{E}\cdot\Delta_q\mathbf{u} - \varepsilon\int\Delta_q(\mathbf{u}\times\mathbf{F})\cdot\Delta_q\mathbf{u},$$

where we have used the fact $\varepsilon(\Delta_q\mathbf{u}\times\overline{\mathbf{B}})\cdot\Delta_q\mathbf{u} = 0$.

On the other hand, multiplying the third equation of Eq. (3.6) by $\frac{1}{\bar{n}}\Delta_q\mathbf{E}$ and the last one by $\frac{1}{\bar{n}}\Delta_q\mathbf{F}$, integrating it over $\mathbf{R}^N$ after adding the resulting equations together implies

$$\frac{1}{2\bar{n}}\frac{d}{dt}\left(\|\Delta_q\mathbf{E}\|_{L^2}^2 + \|\Delta_q\mathbf{F}\|_{L^2}^2\right)$$

$$(3.8) \qquad = \int \Delta_q \mathbf{u} \cdot \Delta_q \mathbf{E} + \frac{1}{\bar{n}} \Delta_q(h(\varrho)\mathbf{u}) \cdot \Delta_q \mathbf{E},$$

where we used the vector analysis formula $\nabla \cdot (\mathbf{f} \times \mathbf{g}) = (\nabla \times \mathbf{f}) \cdot \mathbf{g} - (\nabla \times \mathbf{g}) \cdot \mathbf{f}$.

Combining with the above identities (3.7)-(3.8), with the aid of Cauchy-Schwartz inequality, we get

$$\frac{1}{2}\frac{d}{dt}\Big(\|\Delta_q\varrho\|_{L^2}^2 + \|\Delta_q\mathbf{u}\|_{L^2}^2 + \frac{1}{\bar{n}}\|\Delta_q\mathbf{E}\|_{L^2}^2 + \frac{1}{\bar{n}}\|\Delta_q\mathbf{F}\|_{L^2}^2\Big) + \frac{1}{\tau}\|\Delta_q\mathbf{u}\|_{L^2}^2$$

$$\leq \frac{1}{2}\|\nabla\mathbf{u}\|_{L^\infty}(\|\Delta_q\varrho\|_{L^2}^2 + \|\Delta_q\mathbf{u}\|_{L^2}^2) + \frac{\gamma-1}{2}\|\nabla\varrho\|_{L^\infty}\|\Delta_q\varrho\|_{L^2}\|\Delta_q\mathbf{u}\|_{L^2}$$

$$+\|[\mathbf{u},\Delta_q]\nabla\varrho\|_{L^2}\|\Delta_q\varrho\|_{L^2} + \|[\mathbf{u},\Delta_q]\cdot\nabla\mathbf{u}\|_{L^2}\|\Delta_q\mathbf{u}\|_{L^2}$$

$$+\frac{\gamma-1}{2}\|[\varrho,\Delta_q]\nabla\varrho\|_{L^2}\|\Delta_q\mathbf{u}\|_{L^2} + \frac{\gamma-1}{2}\|[\varrho,\Delta_q]\mathrm{div}\mathbf{u}\|_{L^2}\|\Delta_q\varrho\|_{L^2}$$

$$(3.9) \qquad +\varepsilon\|\Delta_q(\mathbf{u}\times\mathbf{F})\|_{L^2}\|\Delta_q\mathbf{u}\|_{L^2} + \frac{1}{\bar{n}}\|\Delta_q(h(\varrho)\mathbf{u})\|_{L^2}\|\Delta_q\mathbf{E}\|_{L^2}, \quad t \in [0, T_0].$$

Next, we may neglect the effect of relaxation term $\frac{1}{\tau}\|\Delta_q\mathbf{u}\|_{L^2}^2$, since it is only responsible for the large time behavior of solutions to (3.2)-(3.3). Dividing (3.9) by $\{(\|\Delta_q\varrho\|_{L^2}^2 + \|\Delta_q\mathbf{u}\|_{L^2}^2 + \frac{1}{\bar{n}}\|\Delta_q\mathbf{E}\|_{L^2}^2 + \frac{1}{\bar{n}}\|\Delta_q\mathbf{F}\|_{L^2}^2) + \epsilon\}^{\frac{1}{2}}$ ($\epsilon > 0$ a small quantity), we obtain

$$\frac{1}{2}\frac{d}{dt}\Big\{\Big(\|\Delta_q\varrho\|_{L^2}^2 + \|\Delta_q\mathbf{u}\|_{L^2}^2 + \frac{1}{\bar{n}}\|\Delta_q\mathbf{E}\|_{L^2}^2 + \frac{1}{\bar{n}}\|\Delta_q\mathbf{F}\|_{L^2}^2\Big) + \epsilon\Big\}^{1/2}$$

$$\leq \frac{1}{2}\|\nabla\mathbf{u}\|_{L^\infty}(\|\Delta_q\varrho\|_{L^2} + \|\Delta_q\mathbf{u}\|_{L^2}) + \frac{\gamma-1}{2}\|\nabla\varrho\|_{L^\infty}\|\Delta_q\mathbf{u}\|_{L^2}$$

$$+\|[\mathbf{u},\Delta_q]\nabla\varrho\|_{L^2} + \|[\mathbf{u},\Delta_q]\cdot\nabla\mathbf{u}\|_{L^2} + \frac{\gamma-1}{2}\|[\varrho,\Delta_q]\nabla\varrho\|_{L^2}$$

$$(3.10) \qquad +\frac{\gamma-1}{2}\|[\varrho,\Delta_q]\mathrm{div}\mathbf{u}\|_{L^2} + \varepsilon\|\Delta_q(\mathbf{u}\times\mathbf{F})\|_{L^2} + \frac{1}{\bar{n}}\|\Delta_q(h(\varrho)\mathbf{u})\|_{L^2}$$

for $t \in [0, T_0]$. Integrating (3.10) with respect to the variable t, then taking $\epsilon \to 0$, and using the estimates of commutators and continuity for the composition in the stationary case, see [10], we arrive at

$$2^{q\sigma}\|\Delta_q W\|_{L_t^\infty(L^2)}$$

$$\leq C2^{q\sigma}\|\Delta_q W_0\|_{L^2} + C\int_0^t c_q(\varsigma)\|(\varrho,\mathbf{u},\mathbf{F})\|_{B_{2,1}^\sigma}^2 d\varsigma$$

$$(3.11) \qquad +C\int_0^t 2^{q\sigma}\|(\nabla\varrho,\nabla\mathbf{u})\|_{L^\infty}\|(\Delta_q\varrho,\Delta_q\mathbf{u})\|_{L^2}d\varsigma,$$

where $\|c_q(t)\|_{\ell^1} \leq 1$, for all $t \in [0, T_0]$. Next, summing up (3.11) on $q \geq -1$ gives

$$(3.12) \qquad \|W\|_{\widetilde{L}_t^\infty(B_{2,1}^\sigma)} \leq C\|W_0\|_{B_{2,1}^\sigma} + C\int_0^t \|W(\cdot,\varsigma)\|_{B_{2,1}^\sigma}^2 d\varsigma, \quad t \in [0, T_0].$$

Then it follows from Remark 2.1 and Gronwall's inequality that

$$(3.13) \qquad W \in \widetilde{L}_{T_0}^\infty(B_{2,1}^\sigma).$$

Furthermore, it is just a matter of using the equations (3.2) and Proposition 2.5, we deduce that

$$(3.14) \qquad W_t \in \widetilde{L}_{T_0}^\infty(B_{2,1}^{\sigma-1}).$$

Hence, the proof of Proposition 3.1 is complete. $\square$

3.2. Uniform *a priori* estimate and global existence. In this section, our central task is to derive a crucial (uniform) *a priori* estimate, which enables us to achieve the global existence of classical solutions to (3.2)-(3.3).

PROPOSITION 3.2. *If* $W \in \widetilde{\mathcal{C}}_T(B_{2,1}^\sigma) \cap \widetilde{\mathcal{C}}_T^1(B_{2,1}^{\sigma-1})$ *is a solution of (3.2)-(3.3) for any* $T > 0$ *and* $0 < \tau, \varepsilon \leq 1$. *There exist some positive constants* δ_1, μ_1 *and* C_1 *independent of* (τ, ε) *such that for any* $T > 0$, *if*

$$\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} \leq \delta_1, \tag{3.15}$$

then

$$\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}$$
$$+ \mu_1 \Big\{ \Big\| \Big(\sqrt{\tau} \varrho, \frac{1}{\sqrt{\tau}} \mathbf{u}, \sqrt{\tau \varepsilon} \mathbf{E} \Big) \Big\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \Big\| \frac{1}{\sqrt{\varepsilon}} \nabla \mathbf{F} \Big\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})} \Big\}$$
$$\leq C_1 \|W_0\|_{B_{2,1}^\sigma}. \tag{3.16}$$

Having Proposition 3.2, thanks to the standard continuation argument, we can extend the local-in-time solutions in Proposition 3.1, and achieve the global existence of classical solutions to the system (3.2)-(3.3), here we omit details, see e.g. [10]. It follows from Remark 2.1, Proposition 2.5 and the imbedding property $B_{2,1}^\sigma \hookrightarrow \mathcal{C}^1$ that $W \in \mathcal{C}^1([0, \infty) \times \mathbf{R}^N)$ solves (3.2)-(3.3). The choice of δ_1 is sufficient to ensure $\frac{\gamma-1}{2} \varrho + \bar{\psi} > 0$. Then according to Remark 3.1, we know $(n, \mathbf{u}, \mathbf{E}, \mathbf{B}) \in \mathcal{C}^1([0, \infty) \times \mathbf{R}^N)$ is a solution of (1.1)-(1.3) with $n > 0$. Furthermore, we arrive at Theorem 1.2.

Actually, the proof of Proposition 3.2 is to capture the dissipation rates from contributions of $(\varrho, \mathbf{u}, \mathbf{E}, \mathbf{F})$ in turn by using the high- and low-frequency decomposition methods. To do this, we divide it into several lemmas.

LEMMA 3.3. *If* $W \in \widetilde{\mathcal{C}}_T(B_{2,1}^\sigma) \cap \widetilde{\mathcal{C}}_T^1(B_{2,1}^{\sigma-1})$ *is a solution of (3.2)-(3.3) for any* $T > 0$ *and* $0 < \tau, \varepsilon \leq 1$, *then the following estimate holds:*

$$\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + \sqrt{\frac{\mu_2}{\tau}} \|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$
$$\leq C \|W_0\|_{B_{2,1}^\sigma} + C \sqrt{\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}} \Big\| \Big(\sqrt{\tau} \varrho, \frac{1}{\sqrt{\tau}} \mathbf{u} \Big) \Big\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}, \tag{3.17}$$

where μ_2, C *are some uniform positive constants independent of* (τ, ε).

Proof. By integrating (3.9) with respect to $t \in [0, T]$, with the help of Cauchy-Schwartz inequality, we have

$$\frac{1}{2} \Big(\|\Delta_q \varrho\|_{L^2}^2 + \|\Delta_q \mathbf{u}\|_{L^2}^2 + \frac{1}{\bar{n}} \|\Delta_q \mathbf{E}\|_{L^2}^2 + \frac{1}{\bar{n}} \|\Delta_q \mathbf{F}\|_{L^2}^2 \Big) \Big|_0^t + \frac{1}{\tau} \|\Delta_q \mathbf{u}\|_{L_t^2(L^2)}^2$$
$$\leq \frac{1}{2} \|\nabla \mathbf{u}\|_{L_T^2(L^\infty)} \Big(\|\Delta_q \varrho\|_{L_T^2(L^2)} \|\Delta_q \varrho\|_{L_T^\infty(L^2)} + \|\Delta_q \mathbf{u}\|_{L_T^2(L^2)} \|\Delta_q \mathbf{u}\|_{L_T^\infty(L^2)} \Big)$$
$$+ \frac{\gamma-1}{2} \|\nabla \varrho\|_{L_T^\infty(L^\infty)} \|\Delta_q \varrho\|_{L_T^2(L^2)} \|\Delta_q \mathbf{u}\|_{L_T^2(L^2)} + \|[\mathbf{u}, \Delta_q] \cdot \nabla \varrho\|_{L_T^2(L^2)} \|\Delta_q \varrho\|_{L_T^2(L^2)}$$
$$+ \|[\mathbf{u}, \Delta_q] \cdot \nabla \mathbf{u}\|_{L_T^2(L^2)} \|\Delta_q \mathbf{u}\|_{L_T^2(L^2)} + \frac{\gamma-1}{2} \|[\varrho, \Delta_q] \nabla \varrho\|_{L_T^2(L^2)} \|\Delta_q \mathbf{u}\|_{L_T^2(L^2)}$$
$$+ \frac{\gamma-1}{2} \|[\varrho, \Delta_q] \mathrm{div} \mathbf{u}\|_{L_T^2(L^2)} \|\Delta_q \varrho\|_{L_T^2(L^2)} + \varepsilon \|\Delta_q (\mathbf{u} \times \mathbf{F})\|_{L_T^2(L^2)} \|\Delta_q \mathbf{u}\|_{L_T^2(L^2)}$$

$$(3.18) \quad +\frac{1}{\overline{n}}\|\Delta_q(h(\varrho)\mathbf{u})\|_{L^1_T(L^2)}\|\Delta_q\mathbf{E}\|_{L^\infty_T(L^2)}.$$

There exists a constant $\mu_2 > 0$ independent of (τ, ε) after multiplying the factor $2^{2q\sigma}$ on both sides of (3.18), such that

$$2^{2q\sigma}\|\Delta_q W\|^2_{L^2} + \frac{\mu_2}{\tau}2^{2q\sigma}\|\Delta_q\mathbf{u}\|^2_{L^2_t(L^2)}$$

$$\leq C2^{2q\sigma}\|\Delta_q W_0\|^2_{L^2}$$

$$+C\Big\{\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big(c_q^2\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})} + c_q^2\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\mathbf{u}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}\Big)$$

$$+c_q^2\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + c_q^2\|\mathbf{u}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}\|\mathbf{u}\|^2_{\widetilde{L}^2_T(B^\sigma_{2,1})}$$

$$(3.19) \quad +c_q^2\|\mathbf{u}\times\mathbf{F}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + c_q^2\|h(\varrho)\mathbf{u}\|_{\widetilde{L}^1_T(B^\sigma_{2,1})}\|\mathbf{E}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}\Big\},$$

where we used Remark 2.1, Lemma 2.6 and the smallness of $\varepsilon(0 < \varepsilon \leq 1)$; Here and below $C > 0$ denotes a uniform constant independent of (τ, ε); $\{c_q\}$ denotes some sequence which satisfies $\|(c_q)\|_{l^1} \leq 1$ although each $\{c_q\}$ is possibly different in (3.19).

Then, with aid of Young's inequality($\sqrt{fg} \leq (f + g)/2,\ f, g \geq 0$), it follows from Proposition 2.5 and Proposition 2.7 that

$$2^{q\sigma}\|\Delta_q W\|_{L^\infty_T(L^2)} + \sqrt{\frac{\mu_2}{\tau}}2^{q\sigma}\|\Delta_q\mathbf{u}\|_{L^2_T(L^2)}$$

$$\leq C2^{q\sigma}\|\Delta_q W_0\|_{L^2}$$

$$+Cc_q\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\sqrt{\tau}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big)$$

$$+Cc_q\sqrt{\|\mathbf{u}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + Cc_q\sqrt{\|\mathbf{F}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}$$

$$(3.20) \quad +Cc_q\sqrt{\|\mathbf{E}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\sqrt{\tau}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big).$$

Hence, summing up (3.20) on $q \geq -1$ yields

$$\|W\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})} + \sqrt{\frac{\mu_2}{\tau}}\|\mathbf{u}\|_{L^2_T(B^\sigma_{2,1})}$$

$$\leq C\|W_0\|_{B^\sigma_{2,1}} + C\sqrt{\|W\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\|(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u})\|_{\widetilde{L}^2_T(B^\sigma_{2,1})},$$

which is just the desired inequality (3.17). $\square$

LEMMA 3.4. *If* $W \in \widetilde{\mathcal{C}}_T(B^\sigma_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{\sigma-1}_{2,1})$ *is a solution of (3.2)-(3.3) for any* $T > 0$ *and* $0 < \tau, \varepsilon \leq 1$, *then the following estimate holds:*

$$\sqrt{\tau}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}$$

$$\leq C(\|(\varrho, \mathbf{u})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})} + \|(\varrho_0, \mathbf{u}_0)\|_{B^\sigma_{2,1}})$$

$$(3.21) \quad +C\Big\{\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \sqrt{\|(\varrho, \mathbf{u}, \mathbf{F})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big\|\Big(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}\Big)\Big\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big\}.$$

where C *is a uniform positive constant independent of* (τ, ε).

Proof. Set

$$W_I = \begin{pmatrix} \varrho \\ \mathbf{u} \end{pmatrix}, \ A_j^I(\mathbf{u}) = \begin{pmatrix} u^j & \bar{\psi}e_j^\top \\ \bar{\psi}e_j & u^j I_{N\times N} \end{pmatrix}$$

($I_{N\times N}$ denotes the unit matrix of order N

and e_j is N-dimensional vector where the jth component is one, others are zero).

Then the first two equations of (3.2) for W_I can be written as the following vector form

$$(3.22) \qquad \partial_t W_I + \sum_{j=1}^{N} A_j^I(\mathbf{u})\partial_{x_j} W_I = \begin{pmatrix} -\frac{\gamma-1}{2}\varrho\mathrm{div}\mathbf{u} \\ -\frac{\mathbf{u}}{\tau} - \frac{\gamma-1}{2}\varrho\nabla\varrho + \mathbf{G} \end{pmatrix},$$

where $\mathbf{G} := -(\mathbf{E} + \varepsilon\mathbf{u} \times (\mathbf{F} + \overline{\mathbf{B}}))$.

To capture the dissipation rate of ϱ, we make the best use of Shizuta-Kawashima skew-symmetric condition in Fourier spaces, which was developed for general hyperbolic systems of balance laws [18, 29]. Thanks to the isentropic Euler equations (3.22), the concrete information of skew-symmetry matrix $K(\xi)$ is well known (*e.g.* see [5]), which is very helpful to estimate the coupled electromagnetic field $(\mathbf{E}, \mathbf{B})$. Now we state the structural condition.

LEMMA 3.5 (Shizuta-Kawashima). *For all $\xi \in \mathbf{R}^N$, $\xi \neq 0$, there exists a real skew-symmetric smooth matrix $K(\xi)$ which is defined in the unit sphere $\mathbf{S}^{N-1}$:*

$$(3.23) \qquad K(\xi) = \begin{pmatrix} 0 & \frac{\xi^\top}{|\xi|} \\ -\frac{\xi}{|\xi|} & 0 \end{pmatrix},$$

such that

$$(3.24) \qquad K(\xi)\sum_{j=1}^{N}\xi_j A_j^I(0) = \begin{pmatrix} \bar{\psi}|\xi| & 0 \\ 0 & -\bar{\psi}\frac{\xi\otimes\xi}{|\xi|} \end{pmatrix},$$

where A_j^I is the matrix appearing in the system (3.22).

First, we rewrite (3.22) into the linearized form

$$(3.25) \qquad \partial_t W_I + \sum_{j=1}^{N} A_j^I(0)\partial_{x_j} W_I = \mathcal{G} + \begin{pmatrix} -\frac{\gamma-1}{2}\varrho\mathrm{div}\mathbf{u} \\ -\frac{\mathbf{u}}{\tau} - \frac{\gamma-1}{2}\varrho\nabla\varrho + \mathbf{G} \end{pmatrix},$$

where

$$(3.26) \qquad \mathcal{G} = \sum_{j=1}^{N}\left\{A_j^I(0) - A_j^I(\mathbf{u})\right\}\partial_{x_j} W_I.$$

Applying the operator Δ_q to the system (3.25) gives

$$\partial_t \Delta_q W_I + \sum_{j=1}^{N} A_j^I(0)\partial_{x_j}\Delta_q W_I$$

$$(3.27) \qquad = \Delta_q \mathcal{G} + \begin{pmatrix} -\frac{\gamma-1}{2}\Delta_q(\varrho\mathrm{div}\mathbf{u}) \\ -\frac{\Delta_q\mathbf{u}}{\tau} - \frac{\gamma-1}{2}\Delta_q(\varrho\nabla\varrho) + \Delta_q\mathbf{G} \end{pmatrix}.$$

Then we perform the Fourier transform (in the space variable x) for (3.27), multiply the resulting equation by $-i\tau(\widehat{\Delta_q W_I})^* K(\xi)$(* represents transpose and conjugator), and take the real part of each term in the equality. Using the expression (3.23) of the matrix $K(\xi)$ we obtain

$$\tau \text{Im}\Big((\widehat{\Delta_q W_I})^* K(\xi)\frac{d}{dt}\widehat{\Delta_q W_I}\Big) + \tau(\widehat{\Delta_q W_I})^* K(\xi)\Big(\sum_{j=1}^N \xi_j A_j^I(0)\Big)\widehat{\Delta_q W_I}$$

$$= \tau \text{Im}\Big((\widehat{\Delta_q W_I})^* K(\xi)(\widehat{\Delta_q \mathcal{G}})\Big) - \text{Im}\Big((\overline{\widehat{\Delta_q \varrho}})\frac{\xi^\top}{|\xi|}\widehat{\Delta_q \mathbf{u}}\Big) + \tau \text{Im}\Big((\overline{\widehat{\Delta_q \varrho}})\frac{\xi^\top}{|\xi|}\widehat{\Delta_q \mathbf{G}}\Big)$$

$$(3.28) \quad + \frac{\gamma-1}{2}\tau\text{Im}\Big(\overline{\widehat{\Delta_q \mathbf{u}}}\cdot\frac{\xi}{|\xi|}(\Delta_q(\widehat{\varrho\text{div}\mathbf{u}}))\Big) - \frac{\gamma-1}{2}\tau\text{Im}\Big(\overline{\widehat{\Delta_q \varrho}}\frac{\xi^\top}{|\xi|}(\Delta_q(\widehat{\varrho\nabla\varrho}))\Big).$$

The skew-symmetry of $K(\xi)$ implies the relation

$$(3.29) \qquad \text{Im}\Big((\widehat{\Delta_q W_I})^* K(\xi)\frac{d}{dt}\widehat{\Delta_q W_I}\Big) = \frac{1}{2}\frac{d}{dt}\text{Im}\Big((\widehat{\Delta_q W_I})^* K(\xi)\widehat{\Delta_q W_I}\Big).$$

Substituting (3.24) into the second term on the left-hand side of (3.28), it is not difficult to get a lower bound. Indeed, we have

$$\tau \text{Im}\Big((\widehat{\Delta_q W_I})^* K(\xi)\frac{d}{dt}\widehat{\Delta_q W_I}\Big) + \tau(\widehat{\Delta_q W_I})^* K(\xi)\Big(\sum_{j=1}^N \xi_j A_j^I(0)\Big)\widehat{\Delta_q W_I}$$

$$(3.30) \quad \geq \frac{\tau}{2}\frac{d}{dt}\text{Im}\Big((\widehat{\Delta_q W_I})^* K(\xi)\widehat{\Delta_q W_I}\Big) + \bar{\psi}\tau|\xi||\widehat{\Delta_q W_I}|^2 - 2\bar{\psi}\tau|\xi||\widehat{\Delta_q \mathbf{u}}|^2.$$

With the help of Young inequality and the uniform boundedness of the matrix $K(\xi)(\xi \neq 0)$, the right-side of (3.28) can be estimated as

$$\tau \text{Im}\Big((\widehat{\Delta_q W_I})^* K(\xi)(\widehat{\Delta_q \mathcal{G}})\Big) - \text{Im}\Big((\overline{\widehat{\Delta_q \varrho}})\frac{\xi^\top}{|\xi|}\widehat{\Delta_q \mathbf{u}}\Big) + \tau \text{Im}\Big((\overline{\widehat{\Delta_q \varrho}})\frac{\xi^\top}{|\xi|}\widehat{\Delta_q \mathbf{G}}\Big)$$

$$+ \frac{\gamma-1}{2}\tau\text{Im}\Big(\overline{\widehat{\Delta_q \mathbf{u}}}\cdot\frac{\xi}{|\xi|}(\Delta_q(\widehat{\varrho\text{div}\mathbf{u}}))\Big) - \frac{\gamma-1}{2}\tau\text{Im}\Big(\overline{\widehat{\Delta_q \varrho}}\frac{\xi^\top}{|\xi|}(\Delta_q(\widehat{\varrho\nabla\varrho}))\Big)$$

$$\leq \frac{\bar{\psi}\tau}{2}|\xi||\widehat{\Delta_q W_I}|^2 + \frac{C}{\tau|\xi|}|\widehat{\Delta_q \mathbf{u}}|^2 + \tau|\widehat{\Delta_q W_I}||\widehat{\Delta_q \mathcal{G}}| + C\tau|\widehat{\Delta_q \mathbf{u}}||(\Delta_q(\widehat{\varrho\text{div}\mathbf{u}}))|$$

$$(3.31) \quad + C\tau|\widehat{\Delta_q \varrho}||(\Delta_q(\widehat{\varrho\nabla\varrho}))| + \tau\text{Im}\Big((\overline{\widehat{\Delta_q \varrho}})\frac{\xi^\top}{|\xi|}\widehat{\Delta_q \mathbf{G}}\Big),$$

where $C > 0$ is a constant independent of (τ, ε). Combining the equality (3.28) and the inequality (3.30)-(3.31), we deduce that

$$\frac{\bar{\psi}\tau}{2}|\xi||\widehat{\Delta_q W_I}|^2$$

$$\leq \frac{C}{\tau}\Big(|\xi| + \frac{1}{|\xi|}\Big)|\widehat{\Delta_q \mathbf{u}}|^2 + \tau|\widehat{\Delta_q W_I}||\widehat{\Delta_q \mathcal{G}}|$$

$$+ C\tau|\widehat{\Delta_q \mathbf{u}}||(\Delta_q(\widehat{\varrho\text{div}\mathbf{u}}))| + C\tau|\widehat{\Delta_q \varrho}||(\Delta_q(\widehat{\varrho\nabla\varrho}))|$$

$$(3.32) \qquad + \tau\text{Im}\Big((\overline{\widehat{\Delta_q \varrho}})\frac{\xi^\top}{|\xi|}\widehat{\Delta_q \mathbf{G}}\Big) - \frac{\tau}{2}\frac{d}{dt}\text{Im}\Big((\widehat{\Delta_q W_I})^* K(\xi)\widehat{\Delta_q W_I}\Big).$$

Multiplying (3.32) by $|\xi|$ and integrating it over $[0, t] \times \mathbf{R}^N$, then using Plancherel's theorem yields

$$\frac{\bar{\psi}\tau}{2} \int_0^t \|\Delta_q \nabla W_I\|_{L^2}^2 d\varsigma$$

$$\leq \frac{C}{\tau} \int_0^t (\|\Delta_q \mathbf{u}\|_{L^2}^2 + \|\Delta_q \nabla \mathbf{u}\|_{L^2}^2) d\varsigma + C\tau \int_0^t \|\Delta_q \nabla W_I\|_{L^2} \|\Delta_q \mathcal{G}\|_{L^2} d\varsigma$$

$$+ C\tau \int_0^t \|\Delta_q \nabla \mathbf{u}\|_{L^2} \|\Delta_q (\varrho \mathrm{div} \mathbf{u})\|_{L^2} d\varsigma + C\tau \int_0^t \|\Delta_q \nabla \varrho\|_{L^2} \|\Delta_q (\varrho \nabla \varrho)\|_{L^2} d\varsigma$$

$$(3.33) \quad + \tau \int_0^t \mathrm{Im} \int \left((\overline{\widehat{\Delta_q \varrho}}) \xi^\top \widehat{\Delta_q \mathbf{G}} \right) d\xi d\varsigma - \frac{\tau}{2} \mathrm{Im} \int |\xi| \left((\widehat{\Delta_q W_I})^* K(\xi) \widehat{\Delta_q W_I} \right) d\xi \Big|_0^t.$$

The matrix $K(\xi)$ is uniform bounded when $\xi \in \mathbf{R}^N (\xi \neq 0)$, thus we have

$$-\frac{\tau}{2} \mathrm{Im} \int |\xi| \left((\widehat{\Delta_q W_I})^* K(\xi) \widehat{\Delta_q W_I} \right) d\xi \Big|_0^t$$

$$\leq C\tau \left(\int (1 + |\xi|^2) |\widehat{\Delta_q W_I}(t)|^2 d\xi + \int (1 + |\xi|^2) |\widehat{\Delta_q W_I}(0)|^2 d\xi \right)$$

$$(3.34) \leq C(\|\Delta_q W_I(t)\|_{L^2}^2 + \|\Delta_q \nabla W_I(t)\|_{L^2}^2 + \|\Delta_q W_I(0)\|_{L^2}^2 + \|\Delta_q \nabla W_I(0)\|_{L^2}^2),$$

where we used the smallness of $\tau(0 < \tau \leq 1)$ in the last step.

Next we turn to estimate the coupled electromagnetic field $(\mathbf{E}, \mathbf{B})$:

$$\tau \int_0^t \mathrm{Im} \int \left((\overline{\widehat{\Delta_q \varrho}}) \xi^\top \widehat{\Delta_q \mathbf{G}} \right) d\xi d\varsigma$$

$$= -\tau \int_0^t \mathrm{Im} \int \left((\overline{\widehat{\Delta_q \varrho}}) \xi^\top \widehat{\Delta_q \mathbf{E}} \right) d\xi d\varsigma$$

$$- \tau\varepsilon \int_0^t \mathrm{Im} \int \left((\overline{\widehat{\Delta_q \varrho}}) \xi^\top \widehat{\Delta_q (\mathbf{u} \times (\mathbf{F} + \overline{\mathbf{B}}))} \right) d\xi d\varsigma$$

$$(3.35) \qquad \equiv \mathbf{I}_1 + \mathbf{I}_2,$$

where the first term $\mathbf{I}_1$ is estimated as follows

$$\mathbf{I}_1 = -\tau \int_0^t \mathrm{Im} \int \left((\overline{\widehat{\Delta_q \varrho}}) \xi^\top \widehat{\Delta_q \mathbf{E}} \right) d\xi d\varsigma$$

$$= -\tau \left\{ -\int_0^t \frac{i}{2} \int \left((\overline{\widehat{\Delta_q \varrho}}) \xi^\top \widehat{\Delta_q \mathbf{E}} \right) d\xi d\varsigma + \int_0^t \frac{i}{2} \int \left((\widehat{\Delta_q \varrho}) \xi^\top \overline{\widehat{\Delta_q \mathbf{E}}} \right) d\xi d\varsigma \right\}$$

$$= -\frac{\tau}{2} \left\{ \int_0^t \int (\overline{\widehat{\Delta_q \nabla \varrho}}) \cdot \widehat{\Delta_q \mathbf{E}} d\xi d\varsigma + \int_0^t \int (\widehat{\Delta_q \nabla \varrho}) \cdot \overline{\widehat{\Delta_q \mathbf{E}}} d\xi d\varsigma \right\}$$

$$= -\tau (2\pi)^N \int_0^t \int \Delta_q \nabla \varrho \cdot \Delta_q \mathbf{E} dx d\varsigma$$

$$= \tau (2\pi)^N \int_0^t \int \Delta_q \varrho \Delta_q \mathrm{div} \mathbf{E} dx d\varsigma$$

$$= -\tau (2\pi)^N \int_0^t \int \Delta_q \varrho \Delta_q (h(\varrho) - h(0)) dx d\varsigma$$

$$= -\tau (P_0 \gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} (2\pi)^N \int_0^t \|\Delta_q \varrho\|_{L^2}^2 d\varsigma - \tau (2\pi)^N \int_0^t \int \Delta_q \varrho \Delta_q (\tilde{h}(\varrho) \varrho) dx d\varsigma$$

$$(3.36) \quad \leq -\tau(P_0\gamma)^{-\frac{1}{2}}\bar{n}^{\frac{3-\gamma}{2}}(2\pi)^N \int_0^t \|\Delta_q\varrho\|_{L^2}^2 d\varsigma + C\tau \int_0^t \|\Delta_q(\tilde{h}(\varrho)\varrho)\|_{L^2}\|\Delta_q\varrho\|_{L^2} d\varsigma.$$

Here, $\tilde{h}(\varrho) = \int_0^1 h'(\epsilon\varrho)d\varsigma - (P_0\gamma)^{-\frac{1}{2}}\bar{n}^{\frac{3-\gamma}{2}}$ is a smooth function on $\{\varrho | \frac{\gamma-1}{2}\epsilon\varrho + \bar{\psi} > 0, \ \epsilon \in [0,1]\}$ satisfying $\tilde{h}(0) = 0$.

In a similar way, $\mathbf{I}_2$ is estimated as

$$\mathbf{I}_2 = -\tau\varepsilon \int_0^t \mathrm{Im} \int \Big((\overline{\widehat{\Delta_q\varrho}})\xi^\top \Delta_q(\mathbf{u} \times \widehat{(\mathbf{F}+\overline{\mathbf{B}})}) \Big) d\xi d\varsigma$$

$$(3.37) \qquad \leq C\tau\varepsilon \int_0^t \Big(\|\Delta_q\mathrm{div}(\mathbf{u}\times\overline{\mathbf{B}})\|_{L^2} + \|\Delta_q\mathrm{div}(\mathbf{u}\times\mathbf{F})\|_{L^2} \Big)\|\Delta_q\varrho\|_{L^2} d\varsigma.$$

Thus, combining with (3.33)-(3.37), we get

$$\frac{\bar{\psi}\tau}{2} \int_0^t \|\Delta_q\nabla W_I\|_{L^2}^2 d\varsigma + \tau(P_0\gamma)^{-\frac{1}{2}}\bar{n}^{\frac{3-\gamma}{2}}(2\pi)^N \int_0^t \|\Delta_q\varrho\|_{L^2}^2 d\varsigma$$

$$\leq C(\|\Delta_q W_I(t)\|_{L^2}^2 + \|\Delta_q\nabla W_I(t)\|_{L^2}^2 + \|\Delta_q W_I(0)\|_{L^2}^2 + \|\Delta_q\nabla W_I(0)\|_{L^2}^2)$$

$$+\frac{C}{\tau} \int_0^t (\|\Delta_q\mathbf{u}\|_{L^2}^2 + \|\Delta_q\nabla\mathbf{u}\|_{L^2}^2)d\varsigma + C\tau \int_0^t \|\Delta_q\nabla W_I\|_{L^2}\|\Delta_q\mathcal{G}\|_{L^2} d\varsigma$$

$$+C\tau \int_0^t \|\Delta_q\nabla\mathbf{u}\|_{L^2}\|\Delta_q(\varrho\mathrm{div}\mathbf{u})\|_{L^2} d\varsigma + C\tau \int_0^t \|\Delta_q\nabla\varrho\|_{L^2}\|\Delta_q(\varrho\nabla\varrho)\|_{L^2} d\varsigma$$

$$+C\tau \int_0^t \|\Delta_q(\tilde{h}(\varrho)\varrho)\|_{L^2}\|\Delta_q\varrho\|_{L^2} d\varsigma$$

$$(3.38) \quad +C\tau\varepsilon \int_0^t \Big(\|\Delta_q\mathrm{div}(\mathbf{u}\times\overline{\mathbf{B}})\|_{L^2} + \|\Delta_q\mathrm{div}(\mathbf{u}\times\mathbf{F})\|_{L^2} \Big)\|\Delta_q\varrho\|_{L^2} d\varsigma.$$

Recalling Lemma 2.2, we have

$$\|\Delta_q\nabla f\|_{L^2} \approx 2^q\|\Delta_q f\|_{L^2} \ (q \geq 0).$$

Note that this fact, from (3.38), we get the high-frequency part of $\|\Delta_q\varrho\|_{L_t^2(L^2)}(q \geq 0)$:

$$\frac{\bar{\psi}\tau}{2}2^{2q}\|\Delta_q\varrho\|_{L_t^2(L^2)}^2$$

$$\leq C(2^{2q}\|\Delta_q W_I\|_{L_T^\infty(L^2)}^2 + 2^{2q}\|\Delta_q W_I(0)\|_{L^2}^2) + C\Big\{ \frac{2^{2q}}{\tau}\|\Delta_q\mathbf{u}\|_{L_T^2(L^2)}^2$$

$$+\tau\varepsilon\|\Delta_q\varrho\|_{L_T^2(L^2)}\|\Delta_q\mathrm{div}(\mathbf{u}\times\overline{\mathbf{B}})\|_{L_T^2(L^2)} + 2^q\tau\|\Delta_q W_I\|_{L_T^2(L^2)}\|\Delta_q\mathcal{G}\|_{L_T^2(L^2)}$$

$$+2^q\tau\|\Delta_q\mathbf{u}\|_{L_T^2(L^2)}\|\Delta_q(\varrho\mathrm{div}\mathbf{u})\|_{L_T^2(L^2)} + 2^q\tau\|\Delta_q\varrho\|_{L_T^2(L^2)}\|\Delta_q(\varrho\nabla\varrho)\|_{L_T^2(L^2)}$$

$$(3.39) \quad +\tau\|\Delta_q(\tilde{h}(\varrho)\varrho)\|_{L_T^2(L^2)}\|\Delta_q\varrho\|_{L_T^2(L^2)} + \tau\varepsilon\|\Delta_q\mathrm{div}(\mathbf{u}\times\mathbf{F})\|_{L_T^2(L^2)}\|\Delta_q\varrho\|_{L_T^2(L^2)} \Big\}$$

and the corresponding low-frequency part:

$$\tau(P_0\gamma)^{-\frac{1}{2}}\bar{n}^{\frac{3-\gamma}{2}}(2\pi)^N\|\Delta_{-1}\varrho\|_{L_t^2(L^2)}^2$$

$$\leq C(\|\Delta_{-1}W_I\|_{L_T^\infty(L^2)}^2 + \|\widehat{\Delta_{-1}W_I}(0)\|_{L^2}^2)$$

$$+C\Big\{ \frac{1}{\tau}\|\Delta_{-1}\mathbf{u}\|_{L_T^2(L^2)}^2 + \tau\varepsilon\|\Delta_{-1}\varrho\|_{L_T^2(L^2)}\|\Delta_{-1}\mathrm{div}(\mathbf{u}\times\overline{\mathbf{B}})\|_{L_T^2(L^2)}$$

$$+\tau\|\Delta_{-1}W_I\|_{L_T^2(L^2)}\|\Delta_{-1}\mathcal{G}\|_{L_T^2(L^2)} + \tau\|\Delta_{-1}\mathbf{u}\|_{L_T^2(L^2)}\|\Delta_{-1}(\varrho\mathrm{div}\mathbf{u})\|_{L_T^2(L^2)}$$

$$+\tau\|\Delta_{-1}\varrho\|_{L_T^2(L^2)}\|\Delta_{-1}(\varrho\nabla\varrho)\|_{L_T^2(L^2)} + \tau\|\Delta_{-1}(\tilde{h}(\varrho)\varrho)\|_{L_T^2(L^2)}\|\Delta_{-1}\varrho\|_{L_T^2(L^2)}$$

$$(3.40) \quad +\tau\varepsilon\|\Delta_{-1}\mathrm{div}(\mathbf{u}\times\mathbf{F})\|_{L_T^2(L^2)}\|\Delta_{-1}\varrho\|_{L_T^2(L^2)} \Big\}.$$

To conclude, we combine (3.39)-(3.40) and multiply the factor $2^{2q(\sigma-1)}$ on both sides of the resulting inequality to obtain

$$\tau 2^{2q\sigma}\|\Delta_q\varrho\|^2_{L^2_t(L^2)}$$

$$\leq Cc_q^2(\|W_I\|^2_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}+\|W_I(0)\|^2_{B^\sigma_{2,1}})$$

$$+C\Big\{\frac{c_q^2}{\tau}\|\mathbf{u}\|^2_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\tau\varepsilon c_q^2\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\mathrm{div}(\mathbf{u}\times\overline{\mathbf{B}})\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}$$

$$+\tau c_q^2\|W_I\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\mathcal{G}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}+\tau c_q^2\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\varrho\,\mathrm{div}\mathbf{u}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}$$

$$+\tau c_q^2\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big(\|\varrho\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}+\|\tilde{h}(\varrho)\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}$$

$$(3.41)\qquad +\varepsilon\|\mathrm{div}(\mathbf{u}\times\mathbf{F})\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\Big\},$$

where $\{c_q\}$ denotes some sequence which satisfies $\|(c_q)\|_{l^1}\leq 1$.

By employing Young's inequality, we are led to the estimate

$$\sqrt{\tau}2^{q\sigma}\|\Delta_q\varrho\|_{L^2_T(L^2)}$$

$$\leq Cc_q(\|W_I\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}+\|W_I(0)\|_{B^\sigma_{2,1}})$$

$$+C\Big\{\frac{c_q}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau\varepsilon}c_q\sqrt{\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}}$$

$$+c_q\sqrt{\|\mathbf{u}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\sqrt{\tau}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big)$$

$$+\frac{c_q}{\sqrt{\tau}}\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}$$

$$(3.42)\qquad +c_q\sqrt{\|(\varrho,\mathbf{F})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\sqrt{\tau}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big)\Big\}.$$

In the end, with the help of the smallness of (τ,ε), summing up (3.42) on $q\geq -1$ concludes the inequality (3.21) readily. $\square$

LEMMA 3.6. *If* $W\in\widetilde{\mathcal{C}}_T(B^\sigma_{2,1})\cap\widetilde{\mathcal{C}}^1_T(B^{\sigma-1}_{2,1})$ *is a solution of (3.2)-(3.3) for any* $T>0$ *and* $0<\tau,\varepsilon\leq 1$, *then the following estimate holds:*

$$\sqrt{\tau\varepsilon}\|\mathbf{E}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}$$

$$\leq C(\|(\mathbf{u},\mathbf{E},\mathbf{F})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}+\|(\mathbf{u}_0,\mathbf{E}_0,\mathbf{F}_0)\|_{B^\sigma_{2,1}})$$

$$+C\Big\{\Big\|\Big(\sqrt{\tau}\varrho,\frac{\mathbf{u}}{\sqrt{\tau}}\Big)\Big\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\Big\|\frac{\nabla\mathbf{F}}{\sqrt{\varepsilon}}\Big\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}$$

$$(3.43)\quad +\sqrt{\|(\varrho,\mathbf{u},\mathbf{F})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big[\Big\|\Big(\sqrt{\tau}\varrho,\frac{\mathbf{u}}{\sqrt{\tau}},\sqrt{\tau\varepsilon}\mathbf{E}\Big)\Big\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\Big\|\frac{\nabla\mathbf{F}}{\sqrt{\varepsilon}}\Big\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big]\Big\},$$

where $C>0$ *is a uniform constant independent of* (τ,ε).

Proof. A nice "div-curl" construction of Maxwell's equations of (3.2) enables us to obtain the high-frequency part of $\mathbf{E}$. Indeed, by applying Δ_q to both side of $\nabla\cdot\mathbf{E}=-h(\varrho)(q\geq 0)$, integrating it over $\mathbf{R}^N$ after multiplying $\nabla\cdot\Delta_q\mathbf{E}$, in virtue of Hölder's inequality, we obtain

$$\|\nabla\cdot\Delta_q\mathbf{E}\|^2_{L^2}$$

$$(3.44)\qquad \leq C\{(P_0\gamma)^{-\frac{1}{2}}\bar{n}^{\frac{3-\gamma}{2}}\|\Delta_q\varrho\|_{L^2}+\|\Delta_q(\widetilde{h}(\varrho)\varrho)\|_{L^2}\}\|\nabla\cdot\Delta_q\mathbf{E}\|_{L^2},$$

where the function $\widetilde{h}(\varrho)$ is defined by (3.36).

On the other hand, applying $\Delta_q(q \geq 0)$ to the fourth equation of (3.2) and multiplying the resulting equation by $\nabla \times \Delta_q \mathbf{E}$, after integration by parts, yields

$$\|\nabla \times \Delta_q \mathbf{E}\|_{L^2}^2 = -\varepsilon \int \partial_t \Delta_q \mathbf{F} \cdot (\nabla \times \Delta_q \mathbf{E})$$

(3.45)
$$= \varepsilon \int (\nabla \times \partial_t \Delta_q \mathbf{F}) \cdot \Delta_q \mathbf{E}.$$

Substituting the third equation of (3.2) into (3.45), by Cauchy-Schwartz inequality, leads to

$$\|\nabla \times \Delta_q \mathbf{E}\|_{L^2}^2 + \|\nabla \times \Delta_q \mathbf{F}\|_{L^2}^2$$
$$\leq \varepsilon \frac{d}{dt} \int (\nabla \times \Delta_q \mathbf{F}) \cdot \Delta_q \mathbf{E} + \bar{n}\varepsilon \|\Delta_q \mathbf{u}\|_{L^2} \|\Delta_q(\nabla \times \mathbf{F})\|_{L^2}$$
(3.46)
$$+\varepsilon \|\Delta_q(h(\varrho)\mathbf{u})\|_{L^2} \|\Delta_q(\nabla \times \mathbf{F})\|_{L^2}.$$

Combining with (3.44) and (3.46), it follows from the elementary relation

$$\|\nabla \mathbf{f}\|_{L^2} \approx \|\nabla \cdot \mathbf{f}\|_{L^2} + \|\nabla \times \mathbf{f}\|_{L^2}$$

that

$$\|\nabla \Delta_q \mathbf{E}\|_{L^2}^2$$
$$\leq C\{(P_0\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} \|\Delta_q \varrho\|_{L^2} + \|\Delta_q(\widetilde{h}(\varrho)\varrho)\|_{L^2}\} \|\nabla \cdot \Delta_q \mathbf{E}\|_{L^2}$$
$$+\varepsilon \frac{d}{dt} \int (\nabla \times \Delta_q \mathbf{F}) \cdot \Delta_q \mathbf{E} + \bar{n}\varepsilon \|\Delta_q \mathbf{u}\|_{L^2} \|\Delta_q(\nabla \times \mathbf{F})\|_{L^2}$$
(3.47)
$$+\varepsilon \|\Delta_q(h(\varrho)\mathbf{u})\|_{L^2} \|\Delta_q(\nabla \times \mathbf{F})\|_{L^2}.$$

Note that $q \geq 0$, from Lemma 2.2, we further get

$$\tau\varepsilon 2^{2q} \|\Delta_q \mathbf{E}\|_{L^2}^2$$
$$\leq C\tau \|\Delta_q \varrho\|_{L^2}^2 + C\tau\varepsilon \|\Delta_q(\widetilde{h}(\varrho)\varrho)\|_{L^2} 2^q \|\Delta_q \mathbf{E}\|_{L^2}$$
$$+\tau\varepsilon^2 \frac{d}{dt} \int (\nabla \times \Delta_q \mathbf{F}) \cdot \Delta_q \mathbf{E} + C\tau\varepsilon^2 \|\Delta_q \mathbf{u}\|_{L^2} \|\Delta_q(\nabla \times \mathbf{F})\|_{L^2}$$
(3.48)
$$+\tau\varepsilon^2 \|\Delta_q(h(\varrho)\mathbf{u})\|_{L^2} \|\Delta_q(\nabla \times \mathbf{F})\|_{L^2},$$

where the smallness of ε is used. Integrating (3.48) in $t \in [0, T]$ implies

$$\tau\varepsilon 2^{2q} \|\Delta_q \mathbf{E}\|_{L_t^2(L^2)}^2$$
$$\leq \tau\varepsilon^2 2^q \|\Delta_q \mathbf{F}\|_{L_T^\infty(L^2)} \|\Delta_q \mathbf{E}\|_{L_T^\infty(L^2)} + \tau\varepsilon^2 2^q \|\Delta_q \mathbf{F}_0\|_{L^2} \|\Delta_q \mathbf{E}_0\|_{L^2}$$
$$+C\tau \|\Delta_q \varrho\|_{L_T^2(L^2)}^2 + C\tau\varepsilon \|\Delta_q(\widetilde{h}(\varrho)\varrho)\|_{L_T^2(L^2)} 2^q \|\Delta_q \mathbf{E}\|_{L_T^2(L^2)}$$
$$+C\tau\varepsilon^2 \|\Delta_q \mathbf{u}\|_{L_T^2(L^2)} \|\Delta_q(\nabla \times \mathbf{F})\|_{L_T^2(L^2)}$$
(3.49)
$$+\tau\varepsilon^2 \|\Delta_q(h(\varrho)\mathbf{u})\|_{L_T^2(L^2)} \|\Delta_q(\nabla \times \mathbf{F})\|_{L_T^2(L^2)}.$$

On the other hand, the desired low-frequency of $\mathbf{E}$ can be deduced from the Lorentz field in the Euler equations of (3.2). Using the second equation of (3.2), we have

(3.50) $$\mathbf{u}_t + \mathbf{E} = -\bar{\psi}\nabla\varrho - \frac{\mathbf{u}}{\tau} - \mathbf{u} \cdot \nabla\mathbf{u} - \frac{\gamma-1}{2}\varrho\nabla\varrho - \varepsilon\mathbf{u} \times (\mathbf{F} + \overline{\mathbf{B}}).$$

Applying the operator Δ_{-1} to (3.50) implies

$$\partial_t \Delta_{-1}\mathbf{u} + \Delta_{-1}\mathbf{E} = -\bar{\psi}\Delta_{-1}\nabla\varrho - \frac{\Delta_{-1}\mathbf{u}}{\tau} - \Delta_{-1}(\mathbf{u}\cdot\nabla\mathbf{u})$$

$$(3.51) \qquad\qquad -\frac{\gamma-1}{2}\Delta_{-1}(\varrho\nabla\varrho) - \varepsilon\Delta_{-1}(\mathbf{u}\times(\mathbf{F}+\overline{\mathbf{B}})).$$

Multiplying (3.51) by $\tau\varepsilon\Delta_{-1}\mathbf{E}$ and integrating the resulting equation over $\mathbf{R}^N$, we get

$$\tau\varepsilon\frac{d}{dt}\int \Delta_{-1}\mathbf{u}\cdot\Delta_{-1}\mathbf{E} + \tau\varepsilon\|\Delta_{-1}\mathbf{E}\|_{L^2}^2 + \tau\varepsilon\bar{\psi}(P_0\gamma)^{-\frac{1}{2}}\bar{n}^{\frac{3-\gamma}{2}}\|\Delta_{-1}\varrho\|_{L^2}^2$$

$$= \bar{n}\tau\varepsilon\|\Delta_{-1}\mathbf{u}\|_{L^2}^2 + \tau\int\Delta_{-1}\mathbf{u}\cdot\Delta_{-1}(\nabla\times\mathbf{F}) + \tau\varepsilon\int\Delta_{-1}\mathbf{u}\cdot\Delta_q(h(\varrho)\mathbf{u})$$

$$+\tau\varepsilon\bar{\psi}\int\Delta_{-1}(\tilde{h}(\varrho)\varrho)\Delta_{-1}\varrho - \varepsilon\int\Delta_{-1}\mathbf{u}\cdot\Delta_{-1}\mathbf{E}$$

$$-\tau\varepsilon^2\int(\Delta_{-1}\mathbf{u}\times\overline{\mathbf{B}})\cdot\Delta_{-1}\mathbf{E} - \tau\varepsilon\int\Delta_{-1}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta_{-1}\mathbf{E}$$

$$(3.52)\qquad -\frac{\gamma-1}{2}\tau\varepsilon\int\Delta_{-1}(\varrho\nabla\varrho)\cdot\Delta_{-1}\mathbf{E} - \tau\varepsilon^2\int\Delta_{-1}(\mathbf{u}\times\mathbf{F})\cdot\Delta_{-1}\mathbf{E},$$

where we have used the third equation of (3.2). From Cauchy-Schwartz and Young's inequalities, we arrive at

$$\tau\varepsilon\frac{d}{dt}\int\Delta_{-1}\mathbf{u}\cdot\Delta_{-1}\mathbf{E} + \frac{\tau\varepsilon}{4}\|\Delta_{-1}\mathbf{E}\|_{L^2}^2$$

$$\leq C\Big(\frac{1}{\tau}\|\Delta_{-1}\mathbf{u}\|_{L^2}^2 + \tau\|\Delta_{-1}\mathbf{u}\|_{L^2}\|\Delta_{-1}(\nabla\times\mathbf{F})\|_{L^2}\Big)$$

$$+C\tau\varepsilon\Big(\|\Delta_{-1}(h(\varrho)\mathbf{u})\|_{L^2}\|\Delta_{-1}\mathbf{u}\|_{L^2} + \|\Delta_{-1}(\tilde{h}(\varrho)\varrho)\|_{L^2}\|\Delta_{-1}\varrho\|_{L^2}$$

$$+\|\Delta_{-1}(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}\|\Delta_{-1}\mathbf{E}\|_{L^2} + \|\Delta_{-1}(\varrho\nabla\varrho)\|_{L^2}\|\Delta_{-1}\mathbf{E}\|_{L^2}$$

$$(3.53)\qquad +\varepsilon\|\Delta_{-1}(\mathbf{u}\times\mathbf{F})\|_{L^2}\|\Delta_{-1}\mathbf{E}\|_{L^2}\Big).$$

Then integrating (3.53) in $t\in[0,T]$ gives

$$\tau\varepsilon\|\Delta_{-1}\mathbf{E}\|_{L_t^2(L^2)}^2$$

$$\leq C\tau\varepsilon(\|\Delta_{-1}\mathbf{u}\|_{L_T^\infty(L^2)}\|\Delta_{-1}\mathbf{E}\|_{L_T^\infty(L^2)} + \|\Delta_{-1}\mathbf{u}_0\|_{L^2}\|\Delta_{-1}\mathbf{E}_0\|_{L^2})$$

$$+C\Big(\frac{1}{\tau}\|\Delta_{-1}\mathbf{u}\|_{L_T^2(L^2)}^2 + \tau\|\Delta_{-1}\mathbf{u}\|_{L_T^2(L^2)}\|\Delta_{-1}(\nabla\times\mathbf{F})\|_{L_T^2(L^2)}\Big)$$

$$+C\tau\varepsilon\Big(\|\Delta_{-1}(h(\varrho)\mathbf{u})\|_{L_T^2(L^2)}\|\Delta_{-1}\mathbf{u}\|_{L_T^2(L^2)} + \|\Delta_{-1}(\tilde{h}(\varrho)\varrho)\|_{L_T^2(L^2)}$$

$$\cdot\|\Delta_{-1}\varrho\|_{L_T^2(L^2)} + \|\Delta_{-1}(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L_T^2(L^2)}\|\Delta_{-1}\mathbf{E}\|_{L_T^2(L^2)}$$

$$(3.54)\quad +\|\Delta_{-1}(\varrho\nabla\varrho)\|_{L_T^2(L^2)}\|\Delta_{-1}\mathbf{E}\|_{L_T^2(L^2)} + \varepsilon\|\Delta_{-1}(\mathbf{u}\times\mathbf{F})\|_{L_T^2(L^2)}\|\Delta_{-1}\mathbf{E}\|_{L_T^2(L^2)}\Big).$$

Therefore, by combining with the high-frequency estimate (3.49) and low-frequency estimate (3.54), we infer that for $q\geq -1$,

$$\tau\varepsilon 2^{2q}\|\Delta_q\mathbf{E}\|_{L_t^2(L^2)}^2$$

$$\leq C2^{2q}(\|(\Delta_q\mathbf{u},\Delta_q\mathbf{E},\Delta_q\mathbf{F})\|_{L_T^\infty(L^2)}^2 + \|(\Delta_q\mathbf{u}_0,\Delta_q\mathbf{E}_0,\Delta_q\mathbf{F}_0)\|_{L^2}^2)$$

$$+C\Big(\frac{1}{\tau}\|\Delta_q\mathbf{u}\|^2_{L^2_T(L^2)} + \tau\|\Delta_q\varrho\|^2_{L^2_T(L^2)} + \tau\|\Delta_q\mathbf{u}\|_{L^2_T(L^2)}\|\Delta_q(\nabla\times\mathbf{F})\|_{L^2_T(L^2)}\Big)$$

$$+C\tau\varepsilon\Big\{\|\Delta_q(h(\varrho)\mathbf{u})\|_{L^2_T(L^2)}\|\Delta_q\mathbf{u}\|_{L^2_T(L^2)} + \|\Delta_q(\tilde{h}(\varrho)\varrho)\|_{L^2_T(L^2)}\|\Delta_q\varrho\|_{L^2_T(L^2)}$$

$$+\Big(\|\Delta_q(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2_T(L^2)} + \|\Delta_q(\varrho\nabla\varrho)\|_{L^2_T(L^2)} + 2^q\|\Delta_q(\widetilde{h}(\varrho)\varrho)\|_{L^2_T(L^2)}$$

$$+\varepsilon\|\Delta_q(\mathbf{u}\times\mathbf{F})\|_{L^2_T(L^2)}\Big)\|\Delta_q\mathbf{E}\|_{L^2_T(L^2)}$$

$$(3.55)\quad +\tau\varepsilon^2\|\Delta_q(h(\varrho)\mathbf{u})\|_{L^2_T(L^2)}\|\Delta_q(\nabla\times\mathbf{F})\|_{L^2_T(L^2)}\Big\}.$$

By multiplying the factor $2^{2q(\sigma-1)}$ on both sides of (3.55), we gather

$$\tau\varepsilon 2^{2q\sigma}\|\Delta_q\mathbf{E}\|^2_{L^2_t(L^2)}$$

$$\leq Cc_q^2(\|(\mathbf{u},\mathbf{E},\mathbf{F})\|^2_{\widetilde{L}^\infty_T(B^\sigma_{2,1})} + \|(\mathbf{u}_0,\mathbf{E}_0,\mathbf{F}_0)\|^2_{B^\sigma_{2,1}})$$

$$+C\Big\{\frac{c_q^2}{\tau}\|\mathbf{u}\|^2_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \tau c_q^2\|\varrho\|^2_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \tau c_q^2\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\nabla\times\mathbf{F}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}$$

$$+\tau\varepsilon c_q^2\|h(\varrho)\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \tau\varepsilon c_q^2\|\tilde{h}(\varrho)\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}$$

$$+\tau\varepsilon c_q^2\Big(\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})} + \|\varrho\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})} + \|\tilde{h}(\varrho)\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}$$

$$(3.56)\quad +\varepsilon\|\mathbf{u}\times\mathbf{F}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\|\mathbf{E}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \tau\varepsilon^2 c_q^2\|h(\varrho)\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\nabla\times\mathbf{F}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big\},$$

where $\{c_q\}$ denotes some sequence which satisfies $\|(c_q)\|_{l^1} \leq 1$.

Then it follows from Young's inequality that

$$\sqrt{\tau\varepsilon}2^{q\sigma}\|\Delta_q\mathbf{E}\|_{L^2_T(L^2)}$$

$$\leq Cc_q(\|(\mathbf{u},\mathbf{E},\mathbf{F})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})} + \|(\mathbf{u}_0,\mathbf{E}_0,\mathbf{F}_0)\|_{B^\sigma_{2,1}})$$

$$+C\Big\{\frac{c_q}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \sqrt{\tau}c_q\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \frac{c_q}{\sqrt{\varepsilon}}\|\nabla\times\mathbf{F}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}$$

$$+c_q\sqrt{\|\mathbf{u}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\sqrt{\tau}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big)$$

$$+c_q\sqrt{\|\mathbf{u}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \sqrt{\tau\varepsilon}\|\mathbf{E}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big)$$

$$+c_q\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}(\sqrt{\tau}\|\varrho\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \sqrt{\tau\varepsilon}\|\mathbf{E}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})})$$

$$+c_q\sqrt{\|\mathbf{F}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \sqrt{\tau\varepsilon}\|\mathbf{E}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\Big)\Big\}$$

$$(3.57)\quad +c_q\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \frac{1}{\sqrt{\varepsilon}}\|\nabla\times\mathbf{F}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big),$$

where we has used Proposition 2.5. Finally, we sum up (3.57) on $q \geq -1$ and deduce the inequality (3.43) immediately. $\square$

LEMMA 3.7. *If* $W \in \widetilde{\mathcal{C}}_T(B^\sigma_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{\sigma-1}_{2,1})$ *is a solution of (3.2)-(3.3) for any* $T > 0$ *and* $0 < \tau, \varepsilon \leq 1$, *then the following estimate holds:*

$$\frac{1}{\sqrt{\varepsilon}}\|\nabla\mathbf{F}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}$$

$$\leq C(\|(\mathbf{E},\mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + \|(\mathbf{E}_0,\mathbf{F}_0)\|_{B_{2,1}^\sigma})$$

$$(3.58) \quad + C\Big\{\Big\|\frac{\mathbf{u}}{\sqrt{\tau}}\Big\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \sqrt{\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\Big(\Big\|\frac{\mathbf{u}}{\sqrt{\tau}}\Big\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \Big\|\frac{\nabla\mathbf{F}}{\sqrt{\varepsilon}}\Big\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\Big)\Big\},$$

where $C > 0$ is a uniform positive constant independent of (τ, ε).

Proof. Multiply the third equation of (3.2) by $-\Delta_q(\nabla\times\mathbf{F})(q\geq-1)$ and integrate the resulting quality over $\mathbf{R}^N$. Then integration by parts implies

$$\frac{d}{dt}\int \Delta_q(\nabla\times\mathbf{E})\cdot\Delta_q\mathbf{F} + \frac{1}{\varepsilon}\|\Delta_q(\nabla\times\mathbf{F})\|_{L^2}^2$$

$$= \int \Delta_q(\nabla\times\mathbf{E})\cdot\Delta_q\mathbf{F}_t - \bar{n}\int\Delta_q\mathbf{u}\cdot\Delta_q(\nabla\times\mathbf{F})$$

$$(3.59) \qquad - \int\Delta_q(h(\varrho)\mathbf{u})\cdot\Delta_q(\nabla\times\mathbf{F}).$$

Substituting the fourth equation of (3.2) into the first term of (3.59), by Cauchy-Schwartz inequality, leads to

$$\frac{d}{dt}\int\Delta_q(\nabla\times\mathbf{E})\cdot\Delta_q\mathbf{F} + \frac{1}{\varepsilon}\|\Delta_q(\nabla\times\mathbf{F})\|_{L^2}^2 + \frac{1}{\varepsilon}\|\Delta_q(\nabla\times\mathbf{E})\|_{L^2}^2$$

$$(3.60) \quad \leq \bar{n}\|\Delta_q\mathbf{u}\|_{L^2}\|\Delta_q(\nabla\times\mathbf{F})\|_{L^2} + \|\Delta_q(h(\varrho)\mathbf{u})\|_{L^2}\|\Delta_q(\nabla\times\mathbf{F})\|_{L^2}.$$

Due to the incompressible condition of $\mathbf{F}$ in (3.2), by integrating (3.60) with respect to $t\in[0,T]$, we easily derive

$$\frac{1}{\varepsilon}\|\Delta_q\nabla\mathbf{F}\|_{L_t^2(L^2)}^2$$

$$\leq \|\Delta_q(\nabla\times\mathbf{E})\|_{L_T^\infty(L^2)}\|\Delta_q\mathbf{F}\|_{L_T^\infty(L^2)} + \|\Delta_q(\nabla\times\mathbf{E}_0)\|_{L^2}\|\Delta_q\mathbf{F}_0\|_{L^2}$$

$$\quad + \bar{n}\|\Delta_q\mathbf{u}\|_{L_T^2(L^2)}\|\Delta_q(\nabla\times\mathbf{F})\|_{L_T^2(L^2)}$$

$$(3.61) \quad + \|\Delta_q(h(\varrho)\mathbf{u})\|_{L_T^2(L^2)}\|\Delta_q(\nabla\times\mathbf{F})\|_{L_T^2(L^2)}.$$

Noticing that the regularity of $\mathbf{E}$ in the assumption of Lemma 3.7, we multiply (3.61) by the factor $2^{2q(\sigma-1)}$ to get

$$\frac{1}{\varepsilon}2^{2q(\sigma-1)}\|\Delta_q\nabla\mathbf{F}\|_{L_t^2(L^2)}^2$$

$$\leq C\Big\{c_q^2\|\mathbf{E}\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}\|\mathbf{F}\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma-1})} + c_q^2\|\mathbf{E}_0\|_{B_{2,1}^\sigma}\|\mathbf{F}_0\|_{B_{2,1}^{\sigma-1}}$$

$$\quad + c_q^2\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\nabla\times\mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}$$

$$(3.62) \qquad + c_q^2\|h(\varrho)\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\nabla\times\mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\Big\},$$

where $\{c_q\}$ denotes some sequence which satisfies $\|(c_q)\|_{l^1}\leq 1$.

Furthermore, we apply Young's inequality to (3.62) and obtain

$$\frac{1}{\sqrt{\varepsilon}}2^{q(\sigma-1)}\|\Delta_q\nabla\mathbf{F}\|_{L_T^2(L^2)}$$

$$\leq Cc_q\Big\{\|(\mathbf{E},\mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + \|(\mathbf{E}_0,\mathbf{F}_0)\|_{B_{2,1}^\sigma}\Big\}$$

$$\quad + C\Big\{c_q\sqrt{\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}}\|\nabla\times\mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}$$

$$(3.63) \qquad + c_q\sqrt{\|h(\varrho)\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}}\|\nabla\times\mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\Big\}.$$

Finally, after summing up (3.63) on $q \geq -1$, it follows from Proposition 2.5 that

$$\frac{1}{\sqrt{\varepsilon}}\|\nabla \mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}$$

$$\leq C(\|(\mathbf{E}, \mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + \|(\mathbf{E}_0, \mathbf{F}_0)\|_{B_{2,1}^\sigma}) + C\Big\{\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$

$$+ \sqrt{\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \frac{1}{\sqrt{\varepsilon}}\|\nabla \times \mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\Big)\Big\}.$$

This is just the inequality (3.58). Hence, the proof of Lemma 3.7 is complete. $\square$

Remark 3.2. In the proof of Lemma 3.7, the dissipation rate of $\mathbf{F}$ is not available due to the absence of low-frequency estimate on $\|\Delta_{-1}\mathbf{F}\|_{L_T^2(L^2)}$. This is a key reason that Chemin-Lerner's spaces with critical regularity are first introduced to establish the global existence of uniform classical solutions. Otherwise, we need to add a little regularity in order to ensure that the Besov spaces (in x) are still continuously embedded in $\mathcal{C}^1(\mathbf{R}^N)$ spaces. For the similar details, the reader is referred to [9].

Having these lemmas proved above, the proof of Proposition 3.2 can be finished.

Proof of Proposition 3.2. Combing (3.17), (3.21), (3.43) and (3.58), we end up with

$$\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + K_1\sqrt{\tau}\|\varrho\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \sqrt{\frac{\mu_2}{\tau}}\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$

$$+K_2\sqrt{\tau\varepsilon}\|\mathbf{E}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \frac{K_3}{\sqrt{\varepsilon}}\|\nabla \mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}$$

$$\leq C\|W_0\|_{B_{2,1}^\sigma} + C\sqrt{\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\Big\|\Big(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}\Big)\Big\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$

$$+CK_1\Big\{(\|W_I\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + \|W_I(0)\|_{B_{2,1}^\sigma}) + \frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$

$$+\sqrt{\|(\varrho, \mathbf{u}, \mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\Big\|\Big(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}\Big)\Big\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\Big\}$$

$$+CK_2\Big\{(\|(\mathbf{u}, \mathbf{E}, \mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + \|(\mathbf{u}_0, \mathbf{E}_0, \mathbf{F}_0)\|_{B_{2,1}^\sigma})$$

$$+\Big\|\Big(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}\Big)\Big\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \frac{1}{\sqrt{\varepsilon}}\|\nabla \mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}$$

$$+\sqrt{\|(\varrho, \mathbf{u}, \mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\Big[\Big\|\Big(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}, \sqrt{\tau\varepsilon}\mathbf{E}\Big)\Big\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$

$$+\Big\|\frac{1}{\sqrt{\varepsilon}}\nabla \mathbf{F}\Big\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\Big]\Big\} + CK_3\Big\{(\|(\mathbf{E}, \mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + \|(\mathbf{E}_0, \mathbf{F}_0)\|_{B_{2,1}^\sigma})$$

$$+\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \sqrt{\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$

$$(3.64) \qquad +\frac{1}{\sqrt{\varepsilon}}\|\nabla \mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\Big)\Big\},$$

where K_1, K_2 and K_3 are some uniform positive constants (independent of (τ, ε)) to be determined. In order to eliminate the terms $\|(\varrho, \mathbf{u}, \mathbf{E}, \mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}$, $\|\sqrt{\tau}\varrho\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$, $\|\mathbf{u}/\sqrt{\tau}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$ and $\|\nabla \mathbf{F}/\sqrt{\varepsilon}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}$ arising in the right-hand side of (3.64), we

may confine the constants to the following cases:

$$K_1 \leq \min\left\{\frac{1}{4C}, \frac{\sqrt{\mu_2}}{4C}\right\}, \ K_2 \leq \min\left\{\frac{1}{4C}, \frac{K_1}{2C}, \frac{\sqrt{\mu_2}}{4C}, \frac{K_3}{2C}\right\}, \ K_3 \leq \min\left\{\frac{1}{4C}, \frac{\sqrt{\mu_2}}{4C}\right\}.$$

Furthermore, it is not difficult to obtain

$$\frac{1}{2}\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} + \frac{K_1\sqrt{\tau}}{2}\|\varrho\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \frac{\sqrt{\mu_2}}{4\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$

$$+ K_2\sqrt{\tau\varepsilon}\|\mathbf{E}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \frac{K_3}{2\sqrt{\varepsilon}}\|\nabla\mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}$$

$$\leq C\|W_0\|_{B_{2,1}^\sigma} + C\sqrt{\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\left\|\left(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}\right)\right\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}$$

$$+ CK_1\left\{\|(\varrho_0, \mathbf{u}_0)\|_{B_{2,1}^\sigma} + \sqrt{\|(\varrho, \mathbf{u}, \mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\right.$$

$$\left.\cdot\left\|\left(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}\right)\right\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\right\} + CK_2\left\{\|(\mathbf{u}_0, \mathbf{E}_0, \mathbf{F}_0)\|_{B_{2,1}^\sigma}\right.$$

$$+ \sqrt{\|(\varrho, \mathbf{u}, \mathbf{F})\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\left[\left\|\left(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}, \sqrt{\tau\varepsilon}\mathbf{E}\right)\right\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\right.$$

$$\left.\left.+ \left\|\frac{\nabla\mathbf{F}}{\sqrt{\varepsilon}}\right\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\right]\right\} + CK_3\left\{\|(\mathbf{E}_0, \mathbf{F}_0)\|_{B_{2,1}^\sigma}\right.$$

$$\left.+ \sqrt{\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\left(\frac{1}{\sqrt{\tau}}\|\mathbf{u}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \frac{1}{\sqrt{\varepsilon}}\|\nabla\mathbf{F}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\right)\right\}$$

$$\leq C\|W_0\|_{B_{2,1}^\sigma} + C\sqrt{\|W\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\left\{\left\|\left(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}, \sqrt{\tau\varepsilon}\mathbf{E}\right)\right\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\right.$$

$$\left.+ \left\|\frac{\nabla\mathbf{F}}{\sqrt{\varepsilon}}\right\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\right\}$$

$$(3.65) \quad \leq C\|W_0\|_{B_{2,1}^\sigma} + C\sqrt{\delta_1}\left\{\left\|\left(\sqrt{\tau}\varrho, \frac{1}{\sqrt{\tau}}\mathbf{u}, \sqrt{\tau\varepsilon}\mathbf{E}\right)\right\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \left\|\frac{\nabla\mathbf{F}}{\sqrt{\varepsilon}}\right\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\right\},$$

where we used the *a priori* assumption (3.15) in the last step of (3.65).

Finally, we choose the positive constant δ_1 such that

$$C\sqrt{\delta_1} < \min\left\{\frac{K_1}{2}, \frac{\sqrt{\mu_2}}{4}, K_2, \frac{K_3}{2}\right\},$$

then the inequality (3.16) follows immediately. This finishes the proof of Proposition 3.2 eventually.

3.3. Non-relativistic limit. In this section, we justify the non-relativistic limit of the system (1.1)-(1.3) with $\tau = 1$.

Proof of Theorem 1.4. For any fixed $T > 0$, let $(n^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{E}^\varepsilon, \mathbf{B}^\varepsilon)$ be the global solution of (1.1)-(1.3) given by Theorem 1.2. It follows from the uniform energy estimate (1.11) and Remark 2.1 that

$$(3.66) \qquad\qquad (n^\varepsilon - \bar{n}, \mathbf{u}^\varepsilon) \in L_T^\infty(B_{2,1}^\sigma) \cap L_T^2(B_{2,1}^\sigma),$$

$$(3.67) \qquad\qquad \mathbf{E}^\varepsilon \in L_T^\infty(B_{2,1}^\sigma), \quad \sqrt{\varepsilon}\mathbf{E}^\varepsilon \in L_T^2(B_{2,1}^\sigma),$$

(3.68) $$\mathbf{B}^{\varepsilon} - \overline{\mathbf{B}} \in L_T^{\infty}(B_{2,1}^{\sigma}), \quad \frac{\nabla \mathbf{B}^{\epsilon}}{\sqrt{\varepsilon}} \in L_T^2(B_{2,1}^{\sigma-1}),$$

uniformly in ε. Note that (3.68), we deduce

$$\left\{ \int_0^T \|\nabla \mathbf{B}^{\varepsilon}(t, \cdot)\|_{B_{2,1}^{\sigma-1}}^2 dt \right\}^{1/2} = \sqrt{\varepsilon} \left\{ \int_0^T \left\| \frac{\nabla \mathbf{B}^{\varepsilon}(t, \cdot)}{\sqrt{\varepsilon}} \right\|_{B_{2,1}^{\sigma-1}}^2 dt \right\}^{1/2}$$

(3.69) $$\leq C\sqrt{\varepsilon} \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

That is,

(3.70) $$\{\nabla \mathbf{B}^{\varepsilon}\} \to \mathbf{0} \quad \text{strongly in} \quad L_T^2(B_{2,1}^{\sigma-1}), \quad \text{as} \quad \varepsilon \to 0.$$

Moreover, with the help of (1.1), we have

(3.71) $$(n_t^{\varepsilon}, \mathbf{u}_t^{\varepsilon}) \in L_T^2(B_{2,1}^{\sigma-1}),$$

(3.72) $$\sqrt{\varepsilon}\mathbf{E}_t^{\varepsilon} \in L_T^2(B_{2,1}^{\sigma-1}),$$

uniformly in ε.

According to (3.66)-(3.68) and (3.71)-(3.72), it can be derived from Proposition 2.3 and Aubin-Lions compactness lemma in [26] that there exists some function $(n^0, \mathbf{u}^0, \mathbf{E}^0) \in \mathcal{C}([0, \infty), \bar{n} + B_{2,1}^{\sigma}) \times \mathcal{C}([0, \infty), B_{2,1}^{\sigma}) \times \mathcal{C}([0, \infty), B_{2,1}^{\sigma})$ such that the sequences (up to subsequences) as $\varepsilon \to 0$, it holds that

(3.73) $$\{n^{\varepsilon}\} \to n^0 \quad \text{strongly in} \quad \mathcal{C}([0, T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

(3.74) $$\{\mathbf{u}^{\varepsilon}\} \to \mathbf{u}^0 \quad \text{strongly in} \quad \mathcal{C}([0, T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

(3.75) $$\{\sqrt{\varepsilon}\mathbf{E}^{\varepsilon}\} \to \mathbf{0} \quad \text{strongly in} \quad \mathcal{C}([0, T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

(3.76) $$\{\mathbf{E}^{\varepsilon}\} \rightharpoonup \mathbf{E}^0 \quad \text{weakly}^{\star} \text{ in} \quad L_T^{\infty}(B_{2,1}^{\sigma}),$$

(3.77) $$\{\mathbf{B}^{\varepsilon}\} \rightharpoonup \overline{\mathbf{B}} \quad \text{weakly}^{\star} \text{ in} \quad L_T^{\infty}(B_{2,1}^{\sigma}),$$

for any $T > 0$ and $\delta \in (0, 1)$. Thus, in the system (1.1)-(1.3), the uniform bounded properties (3.66)-(3.68) as well as the convergence properties (3.69) and (3.73)-(3.77) allow us to pass to the limit $\varepsilon \to 0$ in the sense of distributions, which implies that $(n^0, \mathbf{u}^0, \mathbf{E}^0)$ is a global weak solution to the Euler-Poisson equations (1.6) satisfying (1.12). This completes the proof of Theorem 1.4.

3.4. Relaxation limit. In this section, we prove the relaxation limit of (1.1)-(1.3) with $\varepsilon = 1$.

Proof of Theorem 1.5. From the scaled variable transform (1.7) and the uniform energy estimate (1.11) in Theorem 1.2, it is shown that $(n^{\tau}, \mathbf{u}^{\tau}, \mathbf{E}^{\tau}, \mathbf{B}^{\tau})$ is a unique global solution of the system (1.8) and (1.13), furthermore, for any fixed $T > 0$, we have

(3.78) $$n^{\tau} - \bar{n} \in L_T^{\infty}(B_{2,1}^{\sigma}) \cap L_T^2(B_{2,1}^{\sigma}),$$

(3.79) $$\tau\mathbf{u}^\tau \in L_T^\infty(B_{2,1}^\sigma), \quad \mathbf{u}^\tau \in L_T^2(B_{2,1}^\sigma),$$

(3.80) $$\mathbf{E}^\tau \in L_T^\infty(B_{2,1}^\sigma) \cap L_T^2(B_{2,1}^\sigma),$$

(3.81) $$\mathbf{B}^\tau - \overline{\mathbf{B}} \in L_T^\infty(B_{2,1}^\sigma), \quad \frac{\nabla\mathbf{B}^\tau}{\sqrt{\tau}} \in L_T^2(B_{2,1}^{\sigma-1}),$$

uniformly in τ. Similar to (3.69), it follows from (3.81) that

(3.82) $$\{\nabla\mathbf{B}^\tau\} \to \mathbf{0} \quad \text{strongly in} \quad L_T^2(B_{2,1}^{\sigma-1}), \quad \text{as} \quad \tau \to 0.$$

Moreover, from the equations (1.8), we conclude that

(3.83) $$n_t^\tau \in L_T^2(B_{2,1}^{\sigma-1}),$$

(3.84) $$\tau^2\mathbf{u}_t^\tau \in L_T^2(B_{2,1}^{\sigma-1}),$$

(3.85) $$\sqrt{\tau}\mathbf{E}_t^\tau \in L_T^2(B_{2,1}^{\sigma-1}),$$

uniformly in τ.

Together with (3.78)-(3.81) and (3.83)-(3.85), it follows from Proposition 2.3 and Aubin-Lions compactness lemma in [26] that there exists some function $(\mathcal{N}, \mathcal{U}, \mathcal{E}) \in \mathcal{C}([0,\infty), \bar{n} + B_{2,1}^\sigma) \times L^2([0,\infty), B_{2,1}^\sigma) \times \mathcal{C}([0,\infty), B_{2,1}^\sigma)$ such that the sequences (up to subsequences) as $\tau \to 0$, it holds that

(3.86) $$\{n^\tau\} \to \mathcal{N} \quad \text{strongly in} \quad \mathcal{C}([0,T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

(3.87) $$\{\tau^2\mathbf{u}^\tau\} \to \mathbf{0} \quad \text{strongly in} \quad \mathcal{C}([0,T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

(3.88) $$\{\mathbf{u}^\tau\} \rightharpoonup \mathcal{U} \quad \text{weakly in} \quad L_T^2(B_{2,1}^\sigma),$$

(3.89) $$\{\sqrt{\tau}\mathbf{E}^\tau\} \to \mathbf{0} \quad \text{strongly in} \quad \mathcal{C}([0,T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

(3.90) $$\{\mathbf{E}^\tau\} \rightharpoonup \mathcal{E} \quad \text{weakly}^\star \text{ in} \quad L_T^\infty(B_{2,1}^\sigma),$$

(3.91) $$\{\mathbf{B}^\tau\} \rightharpoonup \overline{\mathbf{B}} \quad \text{weakly}^\star \text{ in} \quad L_T^\infty(B_{2,1}^\sigma),$$

for any $T > 0$ and $\delta \in (0,1)$. Thus, the uniform bounded properties (3.78)-(3.81) as well as the convergence properties (3.82) and (3.86)-(3.91) allow us to pass to the limit $\tau \to 0$ in the system (1.8) and (1.13) in the sense of distributions, which implies that $(\mathcal{N}, \mathcal{E})$ is a global weak solution to the drift-diffusion equations (1.9) satisfying (1.14). Hence, the proof of Theorem 1.5 is complete.

3.5. Combined non-relativistic and relaxation limits. In the last section, we perform the combined relativistic and relaxation limits of (1.1)-(1.3).

Proof of Theorem 1.6. Combined with the scaled variable transform (1.7) where the superscript τ is replaced by (τ, ε) and the uniform energy estimate (1.11) in Theorem 1.2, it is obtained that $(n^{(\tau,\varepsilon)}, \mathbf{u}^{(\tau,\varepsilon)}, \mathbf{E}^{(\tau,\varepsilon)}, \mathbf{B}^{(\tau,\varepsilon)})$ is a unique solution of the system (1.10) and (1.13) with the superscript τ replaced by (τ, ε), furthermore, for any fixed $T > 0$, we infer that

$$(3.92) \qquad n^{(\tau,\varepsilon)} - \bar{n} \in L_T^\infty(B_{2,1}^\sigma) \cap L_T^2(B_{2,1}^\sigma),$$

$$(3.93) \qquad \tau \mathbf{u}^{(\tau,\varepsilon)} \in L_T^\infty(B_{2,1}^\sigma), \quad \mathbf{u}^{(\tau,\varepsilon)} \in L_T^2(B_{2,1}^\sigma),$$

$$(3.94) \qquad \mathbf{E}^{(\tau,\varepsilon)} \in L_T^\infty(B_{2,1}^\sigma), \quad \sqrt{\varepsilon}\mathbf{E}^{(\tau,\varepsilon)} \in L_T^2(B_{2,1}^\sigma),$$

$$(3.95) \qquad \mathbf{B}^{(\tau,\varepsilon)} - \overline{\mathbf{B}} \in L_T^\infty(B_{2,1}^\sigma), \quad \frac{\nabla \mathbf{B}^{(\tau,\varepsilon)}}{\sqrt{\tau\varepsilon}} \in L_T^2(B_{2,1}^{\sigma-1}),$$

uniformly in (τ, ε). The relation (3.95) turns out to yield

$$(3.96) \qquad \{\nabla \mathbf{B}^{(\tau,\varepsilon)}\} \to \mathbf{0} \quad \text{strongly in} \quad L_T^2(B_{2,1}^{\sigma-1}), \quad \text{as} \quad \tau, \varepsilon \to 0.$$

Moreover, using the equations (1.10), we get

$$(3.97) \qquad n_t^{(\tau,\varepsilon)} \in L_T^2(B_{2,1}^{\sigma-1}),$$

$$(3.98) \qquad \tau^2 \mathbf{u}_t^{(\tau,\varepsilon)} \in L_T^2(B_{2,1}^{\sigma-1}),$$

$$(3.99) \qquad \sqrt{\tau\varepsilon}\mathbf{E}_t^{(\tau,\varepsilon)} \in L_T^2(B_{2,1}^{\sigma-1}),$$

uniformly in (τ, ε).

As previously, it follows from the standard weak convergence methods and compactness lemma in [26] that there exists some function $(\mathcal{N}, \mathcal{U}, \mathcal{E}) \in \mathcal{C}([0,\infty), \bar{n} + B_{2,1}^\sigma) \times L^2([0,\infty), B_{2,1}^\sigma) \times \mathcal{C}([0,\infty), B_{2,1}^\sigma)$ such that the sequences (up to subsequences) as $\tau, \varepsilon \to 0$, it holds that

$$(3.100) \qquad \{n^{(\tau,\varepsilon)}\} \to \mathcal{N} \quad \text{strongly in} \quad \mathcal{C}([0,T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

$$(3.101) \qquad \{\tau^2 \mathbf{u}^{(\tau,\varepsilon)}\} \to \mathbf{0} \quad \text{strongly in} \quad \mathcal{C}([0,T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

$$(3.102) \qquad \{\mathbf{u}^{(\tau,\varepsilon)}\} \rightharpoonup \mathcal{U} \quad \text{weakly in} \quad L_T^2(B_{2,1}^\sigma),$$

$$(3.103) \qquad \{\sqrt{\tau\varepsilon}\mathbf{E}^{(\tau,\varepsilon)}\} \to \mathbf{0} \quad \text{strongly in} \quad \mathcal{C}([0,T], (B_{2,1}^{\sigma-\delta})_{\text{loc}}),$$

$$(3.104) \qquad \{\mathbf{E}^{(\tau,\varepsilon)}\} \rightharpoonup \mathcal{E} \quad \text{weakly}^\star \text{ in} \quad L_T^\infty(B_{2,1}^\sigma),$$

$$(3.105) \qquad \{\mathbf{B}^{(\tau,\varepsilon)}\} \rightharpoonup \overline{\mathbf{B}} \quad \text{weakly}^\star \text{ in} \quad L_T^\infty(B_{2,1}^\sigma),$$

for any $T > 0$ and $\delta \in (0,1)$. Thus, in the system (1.10) and (1.13), the uniform bounded properties (3.92)-(3.95) as well as the convergence properties (3.96) and (3.100)-(3.105) allow us to pass to the limits $\tau, \varepsilon \to 0$ in the sense of distributions, which implies that $(\mathcal{N}, \mathcal{E})$ is a global weak solution to the drift-diffusion equations (1.9) satisfying (1.15). This concludes the proof of Theorem 1.6.

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